Elementary Theorems Regarding Blue Isocurvature Perturbations

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Blue CDM-photon isocurvature perturbations are attractive in terms of observability and may be typical from the perspective of generic mass relations in supergravity. We present and apply three theorems useful for blue isocurvature perturbations arising from linear spectator scalar fields. In the process, we give a more precise formula for the blue spectrum associated with the axion model of 0904.3800, which can in a parametric corner give a factor of $O(10)$ correction. We explain how a conserved current associated with Peccei-Quinn symmetry plays a crucial role and explicitly plot several example spectra including the breaks in the spectra. We also resolve a little puzzle arising from a naive multiplication of isocurvature expression that sheds light on the gravitational imprint of the adiabatic perturbations on the fields responsible for blue isocurvature fluctuations.

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1. INTRODUCTION

Single scalar field inflationary models generate approximately adiabatic, scale-invariant, and Gaussian primordial density perturbations [2–10]. This is consistent with the Cosmic Microwave Background (CMB) measurements [11–22] and the Large Scale Structure (LSS) observations [23, 24]. However, non-thermal cold dark matter (CDM) scenarios such as axions [25–27] and WIMPZILLAs [28–33] naturally have observable CDM-photon isocurvature perturbations (e.g. [34–49]) since the CDM never thermalizes with the photons. Indeed, it is remarkable that a subdominant dark matter component as small as 10^{-4} of the total dark matter content can leave an experimentally detectable effect through cosmology (see e.g. [50]). Furthermore, isocurvature perturbations are interesting since it can generate rich density perturbation phenomenology. For example, unlike standard single field inflationary scenarios, degrees of freedom responsible for isocurvature perturbations are able to generate large primordial local non-Gaussianities [50–78].

Scale-invariant isocurvature spectrum is well constrained as its power on CMB length scales has to be less than about 3% of the adiabatic power [13, 16, 79–84]. Because the largest scale invariant isocurvature effects on CMB measurements occurs on long length scales (e.g. see the appendix in [85]), one expects scale invariant isocurvature effects to be well hidden in any future observations probing short length scales. However, if the isocurvature spectrum is very blue, then isocurvature effects that are hidden on long length scales may become large effects on short length scales (see e.g. [86–89]). If such strongly blue spectral index isocurvature signal is uncovered in the future, one may ask what one will learn regarding the high energy physics of the isocurvature sector.

One answer to that is given by [1] in which a supersymmetric axion model is constructed giving rise to a blue spectrum. In that work, the phenomenologically relevant axion isocurvature perturbation amplitude δa is assumed to be given by the frozen value $\delta a/\varphi_+$ at horizon crossing where φ_+ is the classical value of the radial field that breaks the Peccei-Quinn (PQ) symmetry during inflation. Unlike in the conventional minimal axion scenario in which the order parameter φ_+ is sitting at its potential minimum during inflation, its value is initially displaced from the minimum and is slowly decreasing towards the stable minimum during inflation. Hence, the assumed frozen value at horizon crossing increases for larger wave vector k modes which leave the horizon later. In that way, a blue spectrum is generated over a k range that depends on the spectral index which controls the amount of time φ_+ takes to settle to its minimum. Furthermore, because supergravity structure generically induces a Hubble scale mass [90] for φ_+, the spectral index can be easily extremely blue. For example, it has been claimed that this scenario allows an isocurvature spectral index of $n = 4$ for $k \in [k_{\text{min}}, k_{\text{max}}]$ specified by $k_{\text{max}}/k_{\text{min}} \sim \exp(10)$.

Partly motivated by this result, we formulate three elementary "theorems" regarding isocurvature perturbations with a very blue spectrum for non-thermal dark matter fields such as the axions with a displaced (possibly time dependent) vacuum expectation value during inflation.[1] Theorem 1 defines a superhorizon conserved quantity for systems possessing an approximate symmetry associated with linearly perturbed system. The merit of this theorem compared to previous discussions of this topic in the literature (see e.g. [91, 92]) is its ability to go beyond the end of inflation and the reheating process. Theorem 2 describes under what averaging conditions that fluid quantities behave as $\delta\chi_{nad}/\chi_0$. This second theorem merely restates what is known in the literature

[1] The proofs are only at the rigor of a typical physics literature.

in the context of the theorem. Theorem 3 describes the computation of the quantum isocurvature perturbations. The merit of theorem 3 compared to the previous discussion in the literature is the explicit canonical quantization in the presence of linearized gravitational constraints. Furthermore, we point out the clear conditions under which the simple analytic estimates are valid.

We then give couple of applications of our theorems. First, we improve on the naive quantization of axions in scenario of [1] and compute $O(1+\frac{1}{(n-4)^2})$ corrections to the spectrum of with spectral index n. In the process, an interesting application of conserved PQ symmetry current is made, which explains how two independent dynamical degrees of freedom behave as a single one during a finite time duration of interest. The theorems also help to set a precise boundary of where the simple analytic computations are invalid. For example, contrary to claims of [1], $n=4$ spectrum cannot be generated in their scenario. Another consequence of understanding the boundary is that if the ratio of axion isocurvature blue power amplitude to the adiabatic power amplitude is at most of the order of a few percent on the largest observable scales and can be described in terms of quantization methods presented in this paper (and implicitly approximated in [1]), most of the cold dark matter must be made of different species. We also illustrate through example plots, phenomenologically interesting parametric corners of the model (having a six dimensional parameter space). Although observable spectra can contain breaks, these break regions typically contain k-space domains for which the simple analytic computation is invalid. We identify how large the expansion rate H during inflation can be in this class of models generating a large blue spectrum. A measurement of tensor-to-scalar ratio at the level of $r=O(10^{-1})$ will disfavor this class of models, at least in its simplest form.

In another application, theorem 3 is used to explain why the isocurvature blue spectrum does not have a simple lower bound suggested by a naive operator product analysis. More explicitly, the isocurvature perturbations are defined to be a contrast of the form $S_\chi \sim C_1\delta\chi - C_2\delta\phi$ where C_i are background field dependent coefficients and $\delta\phi$ is the inflaton field and $\delta\chi$ is the field responsible for the existence of isocurvature perturbations. In other words, the isocurvature field is always dressed with the inflaton sector. The quantum correlator $\langle SS\rangle$ would then naively have a piece that is proportional to $C_2^2\langle\delta\phi\delta\phi\rangle$ coming from the dressing. This piece for a blue spectrum is of order of the adiabatic perturbation power spectrum. If the cross correlation piece does not precisely cancel this piece, $\langle S_\chi S_\chi\rangle$ would be of the order of adiabatic spectrum, leading to a simple lower bound. However, the theorem shows that the power spectrum of S_χ generically behaves independently of the adiabatic spectrum. The more broad lesson encapsulated by theorem 3 is that

the gravitational coupling of $\delta\chi$ to $\delta\phi$ makes $\delta\chi$ grow an inhomogeneity that looks like $\delta\phi$ such that S_χ becomes independent of $\delta\phi$.

The order of the presentation will be as follows. In Sec. 2, three simple theorems and couple of corollaries useful for spectator dark matter isocurvature spectra are presented. In Sec. 3, a couple of applications of the theorems are given. One application corresponds to improving and elucidating the computation of [1]. The second application corresponds to understanding how dressing effects coming from the definition of the isocurvature perturbations do not mix inflaton field quantum fluctuations with the dark matter field quantum fluctuations because of the secular growth imprinting an adiabatic inhomogeneity to the dark matter field. We close with a summary and thoughts on future work to be done in this direction. In the appendix, we collect some results useful for the theorems.

2. USEFUL SIMPLE THEOREMS FOR BLUE ISOCURVATURE MODELS WITH A SLOWLY ROLLING TIME DEPENDENT VEV

2.1. Definitions

In this subsection, we define the language used for our theorems.

Metric and Fourier Conventions Although theorems that we present are gauge invariant, we will have the occasion to use several gauges in our proofs. The Newtonian gauge scalar perturbations will be parameterized as

$$ds^2 = (1+2\Psi^{(N)})dt^2 - a^2(t)(1+2\Phi^{(N)})|d\vec{x}|^2. \tag{1}$$

We consider slow-roll inflaton field φ scenarios in which superhorizon adiabatic perturbations are approximately conserved. Conserved adiabatic curvature perturbations on superhorizon scales is given in Newtonian gauge by the solution [10, 93–95]

$$\Phi^{(N)}(t,\vec{k}) - H\frac{\delta\rho^{(N)}(t,\vec{k})}{\dot{\rho}(t)} = \zeta_{\vec{k}} \equiv \text{constant} \tag{2}$$

where $H \equiv \dot{a}/a$ and $|\vec{k}/a| \ll H$ and we have introduced the Fourier convention

$$Q(t,\vec{k}) = \int d^3x e^{-i\vec{k}\cdot\vec{x}} Q(t,\vec{x}). \tag{3}$$

In the Newtonian gauge, the expansion is manifestly isotropic. Furthermore, $\Phi^{(N)}$ has the intuitive interpretation of being the gravitational potential in the Poisson equation. Because of these prop-

erties, the field equations in the Newtonian gauge are convenient to work with when working with classical equations.

On the contrary, the spatially flat gauge is more useful for quantization during inflation (see e.g. [96]). The scalar metric perturbation convention in spatially flat gauge can be chosen to be

$$ds^2 = (1+2\Psi^{(sf)})dt^2 + a\partial_i F^{(sf)} dt dx^i - a^2(t)|d\vec{x}|^2. \tag{4}$$

As shown in Sec. 2.4, the relevant interaction action derived from solving the gravitational constraints in this gauge consist only of local terms of the fields unlike for the corresponding equations in the Newtonian gauge. Thus, the quantization of fields and the investigation of the subhorizon mode functions are technically simpler in this gauge. Hence, we will employ the spatially flat gauge only for quantization during inflation which establishes the initial conditions for the late time classical equations.

Linear Spectator Isocurvature Field Let linear spectator isocurvature field be defined as a canonically normalized scalar field $\chi = \chi_0(t) + \delta\chi^{(N)}(t,\vec{x})$ for which

$$\delta\rho_\chi^{(N)} \propto \delta\chi^{(N)} + O(\delta\chi^{(N)2}) \tag{5}$$

$$\chi_0^{(N)}(t_{\text{during inflation}}) \gg \frac{H}{2\pi}\Big|_{\text{during inflation}} \tag{6}$$

$$\frac{\delta\rho_\chi^{(N)}}{\delta\rho_{\text{dominant}}^{(N)}} = \frac{\delta T_{\chi\ \ 0}^{(N)0}}{\delta T_{\text{dominant}\ \ 0}^{(N)0}} \ll 1 \tag{7}$$

in Newtonian gauge where the subscript "dominant" corresponds to the energy density component that dominates $T^{(N)0}_{\ \ \ \ 0}$. For example, during inflation, "dominant" corresponds to the label φ while during radiation domination, "dominant" corresponds to the label γ representing the relativistic degrees of freedom. Because we will focus on

$$V_\chi = \frac{1}{2}m^2\chi^2, \tag{8}$$

Eq. (7) translates to

$$\frac{m^2}{H^2}\frac{\chi_0}{M_p} \ll 3\sqrt{2\varepsilon} \tag{9}$$

where ε is the inflationary slow-roll parameter. If the inflaton potential is given as $V_\varphi(\varphi)$, then

$$\varepsilon = \frac{M_p^2}{2}\left(\frac{V_\varphi'(\varphi_0)}{V_\varphi(\varphi_0)}\right)^2 \tag{10}$$

where $\varphi = \varphi_0(t) + \delta\varphi$. The effective expansion parameters are χ_0/M_p and slow-roll parameters of the inflaton field if $m/H \sim O(1)$.

Spectral Conventions The gauge invariant spectrum of linear spectator χ-photon isocurvature perturbations useful for Boltzmann equations is often defined *during radiation dominated universe* through

$$\Delta_{s_\chi}^2(k) \equiv \frac{k^3}{2\pi^2}\int \frac{d^3k'}{(2\pi)^3}\langle \delta_{s_\chi}(t,\vec{k})\delta_{s_\chi}(t,\vec{k}')\rangle \tag{11}$$

$$\delta_{s_\chi} \equiv 3(\zeta_\chi - \zeta_\gamma) \tag{12}$$

$$\zeta_\chi = \Phi^{(N)} + \frac{\delta\rho_\chi^{(N)}|_{\text{background smoothed}}}{3\langle\rho_\chi + P_\chi\rangle_{\text{time}}}, \quad \zeta_\gamma = \Phi^{(N)} + \frac{\delta\rho_\gamma^{(N)}}{3(\rho_\gamma + P_\gamma)} \tag{13}$$

where P_i are pressure quantities corresponding to $-T_i^i$ components of the energy momentum tensor. Here, the "time" average in the denominator of the definition of ζ_χ corresponds to a time average over m^{-1} time scale. The "background smoothed" in the numerator of the definition of ζ_χ corresponds to averaging over m^{-1} time scale all quadratic terms in the background $\chi_0(t)$ appearing in the numerator. The variables ζ_χ and ζ_γ are conserved outside the horizon if the pressure of the constituent is a function only of its energy density. In particular, ζ_γ corresponds to the gauge-invariant curvature perturbation if we assume that radiation behaves as a single component fluid coming from the inflaton decay. For single-field inflation, observational normalization of

$$\begin{aligned}\Delta_\zeta^2(k_0 = 0.05\text{Mpc}^{-1}) &= \Delta_{\zeta_\gamma}^2(k_0) \\ &\approx \frac{V_\varphi(k_0)}{24\pi^2 M_p^4 \varepsilon} \approx 2.4\times 10^{-9}\end{aligned} \tag{14}$$

corresponds to the currently known approximate value of adiabatic curvature perturbation amplitude.

The χ-photon isocurvature spectrum often contain k-space domains which can be parameterized as

$$\Delta_{s_\chi}^2(k) = \Delta_{s_\chi}^2(k_0)\left(\frac{k}{k_0}\right)^{n-1} \tag{15}$$

where n is the spectral index. The isocurvature spectrum is blue when $n > 1$. The primary focus of this paper is regarding spectra for which $n-1 \gtrsim O(0.1)$ which become parametrically insensitive to the inflationary slow roll parameter values of $O(\varepsilon) < 0.02$. As far as the phenomenological bounds are concerned, note that

$$\Delta_s^2(k) = \omega_\chi^2 \Delta_{s_\chi}^2(k) \tag{16}$$

where $\omega_\chi \leq 1$ is the fraction of cold dark matter that is in the χ field as is explained in Appendix B. The current phenomenological bounds on $\Delta_s^2(k)/\Delta_{s_\chi}^2(k)$ for scale invariant power spectrum is approximately a few percent [13, 16, 79–84].

With these definitions and assumptions, we can construct a useful statement that can be used to set classical equation boundary conditions before Eq. (7) breaks down. The most important of the three theorems that will be presented below is theorem three. Note that one of the key merits of the theorem that we are presenting is its applicability connecting computations during inflation to variables during radiation domination.

2.2. Theorem 1: Classically Conserved Isocurvature Quantity

Here is a statement of the first theorem. In slow-roll inflationary scenarios, the linear spectator isocurvature quantity

$$S_\chi(t,\vec{k}) \equiv \frac{2\delta\chi_{nad}}{\chi_0(t)} \tag{17}$$

where

$$\delta\chi_{nad} \equiv \delta\chi^{(G)}(t,\vec{k}) - \delta\chi_{ad}^{(G)}(t,\vec{k}) \tag{18}$$

on superhorizon length scales is approximately conserved as long as χ interaction is dominated by $V_\chi = m^2\chi^2/2$ *and gravity*, anisotropic stress effects can be neglected, and attractor behavior of $\delta\chi_{nad}$ and $\chi_0(t)$ is relevant *during inflation* (i.e. non-pathological boundary conditions are chosen for the homogeneous field) with an expansion rate of H. A sufficient condition for attractor behavior with non-pathological boundary condition is

$$|\dot{\chi}_0(t_{\text{iniital}})| \lesssim m^2\chi_0(t_{\text{initial}})/H$$

$$\nu N_k \gg 1 \tag{19}$$

where

$$\nu \equiv \frac{3}{2}\sqrt{1 - \frac{4}{9}\frac{m^2}{H^2}} \tag{20}$$

and N_k is the number of efolds between the time of k-mode horizon exit and the end of inflation. Here, we have defined

$$\delta\chi_{ad}^{(G)}(t,\vec{k}) \equiv -\zeta_{\vec{k}}\frac{\dot{\chi}_0(t)}{a(t)}\int dt a(t) + \xi^0\partial_0\chi_0(t) \tag{21}$$

where $\xi^0 = 0$ in the Newtonian gauge (i.e. $G = N$) and in any other gauge G is related to the Newtonian gauge coordinates through $x^{(N)\mu} = x^{(G)\mu} + (\xi^0, \delta^{ij}\partial_i\xi)$. Furthermore, S_χ is a *gauge invariant* quantity. Note that we have introduced a factor of 2 in the definition of S_χ for later

convenience. Finally, note that this theorem is formulated at the classical solution level. The error in the conservation coming from the assumption of attractor behavior can be estimated as

$$\text{attractor error } \sim O\left(\exp\left[-2\nu N_k\right]\right) \tag{22}$$

where the coefficient of the error depends on details of initial conditions of both the homogeneous mode and the perturbation mode at the beginning of inflation. The fractional error $O(\mathscr{E})$ in the conservation is approximated to be the terms that are dropped in making this statement:

$$\mathscr{E} = \exp\left[-2\nu N_k\right] + \frac{\delta\rho_\chi^{(N)}}{\delta\rho_{\text{dominant}}^{(N)}}. \tag{23}$$

We also implicitly assume that the post-inflationary cosmological history consists of smoothly connected patches of power-laws.

It is important to note that this theorem makes S_χ conserved independently of the details not stated in the theorem, including some of the details of the end of the inflation, reheating, early radiation domination, and how χ_0 makes the transition from a slow-roll field to a coherently oscillating one. In particular, the classical conservation here is valid even when $\varepsilon \to 1$ at the end of inflation, unlike the spatially flat gauge quantity $\delta\chi^{(sf)}/\chi_0$ which undergoes generically undergoes time evolution at the end of inflation. The conditions stated in the theorem can be understood as a decoupling limit of the isocurvature perturbations, and this theorem establishes a classically conserved quantity in that limit. Note one of the important points for this paper: the numerator $\delta\chi$ and the denominator χ_0 must correspond to the same dynamical degree of freedom that responds to the same potential V_χ dominated by the mass term. Finally, note that when we state the assumption that the mass term and gravity dominate the interactions, we are stating that perturbative interactions are too weak to thermalize the system.

proof

Consider the equation of motion for the perturbation variable in the Newtonian gauge $\delta\chi^{(N)}$ in the long wavelength limit in which we can neglect the gradient terms:

$$\delta\ddot{\chi}^{(N)} + 3H\delta\dot{\chi}^{(N)} + V_\chi''(\chi_0)\delta\chi^{(N)} - 4\dot{\chi}_0\dot{\Psi}^{(N)} + 2V_\chi'(\chi_0)\Psi^{(N)} = 0. \tag{24}$$

Here, we have assumed that gravitational interactions and potential self-interactions $V_\chi'(\chi_0)$ dominate the interactions. If anisotropic stress effects can be neglected, the ij component of Einstein equations imply

$$\Phi^{(N)} = -\Psi^{(N)}. \tag{25}$$

The 00 component of Einstein equation in Newtonian gauge partially determining $\Psi^{(N)}$ is

$$-3\frac{\dot{a}}{a}(H\Psi^{(N)}+\dot{\Psi}^{(N)})=\frac{1}{2M_p^2}\left[\delta\rho_\chi^{(N)}+\delta\rho_{\text{dominant}}^{(N)}\right] \tag{26}$$

where

$$\delta\rho_\chi^{(N)}=-\dot{\chi}_0^2\Psi+\dot{\chi}_0\partial_t\delta\chi+V_\chi'\delta\chi \tag{27}$$

and $\delta\rho_{\text{dominant}}^{(N)}\equiv\delta T^{0\,(N)}_{\text{dominant}\;0}$ is the dominant contribution to the energy-momentum tensor as discussed in Eq. (7).[2] In the limit

$$\frac{\delta\rho_\chi^{(N)}}{\delta\rho_{\text{dominant}}^{(N)}}\ll 1, \tag{28}$$

we see that $\Psi^{(N)}$ is independently of $\delta\chi$. Hence, with the condition of Eq. (28), the $\Psi^{(N)}$ dependent terms in Eq. (24) are external sources terms. Due to dilatation diffeomorphism gauge solution that lifts to physical solutions [97], there exists an adiabatic solution

$$\delta\chi_{ad}^{(N)}=-\zeta_{\vec{k}}\frac{\dot{\chi}_0(t)}{a(t)}\int dta(t) \tag{29}$$

where $\zeta_{\vec{k}}$ is the usual time independent gauge-invariant curvature perturbation constant determined by the inflaton sector φ approximately independently of $\delta\chi^{(N)}$ as long as Eq. (28) is satisfied. From the perspective of the classical equations we are discussing here, ζ_k is simply a constant parameterizing a solution to Eq. (24) where the gravitational potential is given by Eq. (2).[3]

Given that Eq. (24) is a second order differential equation, the most general solution corresponds to two independent solutions $h_{1,2}$ to the homogeneous equation added to the particular solution given by Eq. (29):

$$\delta\chi^{(N)}=c_1h_1^{(N)}+c_2h_2^{(N)}+\delta\chi_{ad}^{(N)} \tag{30}$$

where c_1 and c_2 are coefficients independent of time. Hence, we see that in the limit that $k/(aH)\to 0$ can be neglected, the numerator of

$$\frac{\delta\chi_{nad}}{\chi_0(t)}=\frac{\delta\chi^{(N)}-\delta\chi_{ad}^{(N)}}{\chi_0(t)}=\frac{c_1(\vec{k})h_1^{(N)}(t,\vec{k})+c_2(\vec{k})h_2^{(N)}(t,\vec{k})}{\chi_0(t)} \tag{31}$$

[2] These statements can easily be covariantized, but such formalizations tend to obscure the intuition rather than to illuminate the intuition. Since our aim is to illuminate the intuition of the simple physics, we will leave the presentation in the explicitly gauge dependent form.

[3] Although we have not made any explicit assumptions about the background energy density, Eq. (28) does depend on the background energy density.

is governed by the same equation as the denominator if $V_\chi = m^2\chi^2/2$: i.e.

$$\ddot{h}_i^{(N)} + 3H\dot{h}_i^{(N)} + m^2 h_i^{(N)} = 0 \tag{32}$$

$$\ddot{\chi}_0^{(N)} + 3H\dot{\chi}_0^{(N)} + m^2\chi_0^{(N)} = 0. \tag{33}$$

Hence, we can also write

$$\chi_0 = e_1 h_1^{(N)} + e_2 h_2^{(N)}. \tag{34}$$

We know that one mode decays faster than the other during inflation. This is what we usually call the attractor behavior during inflation [95]. We will call the less decaying mode $h_1^{(N)}$. More quantitatively, in the dS approximation, we have

$$\left|\frac{h_1^{(N)}}{h_2^{(N)}}\right| = e^{2H\nu t}. \tag{35}$$

In this case, we thus have at the end of inflation (when this relative growth ends)

$$\frac{\delta\chi^{(N)} - \delta\chi_{ad}^{(N)}}{\chi_0(t)} = \frac{c_1(\vec{k}) + c_2(\vec{k})O(e^{-2\nu N_k})}{e_1 + e_2(\vec{k})O(e^{-2\nu N_k})} \tag{36}$$

which is independent of time N_k (the number of scale factor efolds between k mode horizon exit and) as long as

$$\nu N_k \gg 1. \tag{37}$$

Hence, the error in the conservation coming from the attractor assumption is $O(\exp[-2\nu N_k])$.

Finally, under the gauge transformation $x^{(N)\mu} = x^{(G)\mu} + (\xi^0, \delta^{ij}\partial_i\xi)$ we have

$$\begin{aligned} \delta\chi^{(G)}(t,\vec{k}) &= \delta\chi^{(N)}(t,\vec{k}) + \xi^0\partial_0\chi_0(t) && (38)\\ \delta\chi_{ad}^{(G)}(t,\vec{k}) &= \delta\chi_{ad}^{(N)}(t,\vec{k}) + \xi^0\partial_0\chi_0(t) && (39) \end{aligned}$$

on long wavelengths. This means

$$\frac{\delta\chi^{(N)} - \delta\chi_{ad}^{(N)}}{\chi_0(t)} = \frac{\delta\chi^{(G)} - \delta\chi_{ad}^{(G)}}{\chi_0(t)} \tag{40}$$

for general gauges G non-singularly connected to the Newtonian gauge N. We thus see that this quantity is gauge invariant.

Note that one may wonder whether there are other interactions besides mass interactions that would lead to the same result. To see that this is not generically possible with only potential modifications, note that for $\delta\chi^{(N)} - \delta\chi_{ad}^{(N)}$ to behave similarly as $\chi_0(t)$, a generic condition is

$$V_\chi'(\delta\chi) \approx V_\chi''(\delta\chi)\delta\chi \tag{41}$$

which can easily be solved to obtain

$$V_\chi(\delta\chi) \approx C_1 \delta\chi^2 + C_2 \tag{42}$$

which means that χ interaction is dominated by the mass term.

A trivial corollary of this theorem is to discuss the situation when the constant m is replaced by $m(t)$ which is constant during a finite time interval during inflation and makes a transition to another value during inflation.

corollary 1 In the context of theorem 1, suppose m is not a constant but makes a transition to another value during inflation:

$$m^2(t) = \begin{cases} m_1 & t < t_c \\ m_2 & t > t_c \end{cases} \tag{43}$$

where the transition time region near $t = t_c$ is assumed to be much smaller in time than m_1^{-1} and H^{-1}. The quantity S_χ is still conserved as long as sufficient time has passed during the $t < t_c$ period to be in the attractor approximation just as in theorem 1: i.e.

$$H(t_c - t_k)\nu(m_1) \gg 1 \tag{44}$$

where

$$\nu(m_1) = \frac{3}{2}\sqrt{1 - \frac{4}{9}\frac{m_1^2}{H^2}}. \tag{45}$$

The error estimate associated with this conservation is $O(\mathscr{E})$ where

$$\mathscr{E} \equiv \max\left\{ \exp\left[-2\nu(m_1)(t_c - t_k)H\right], \, \frac{\delta\rho_\chi^{(N)}}{\delta\rho_{\text{dominant}}^{(N)}} \right\} \tag{46}$$

again with the neglect of any possible secular effects that depend on unusual cosmological histories.

proof

On superhorizon scales, we have just as in theorem 1 proof

$$\ddot{h}_i^{(N)} + 3H\dot{h}_i^{(N)} + m^2(t)h_i^{(N)} = 0 \tag{47}$$

$$\ddot{\chi}_0^{(N)} + 3H\dot{\chi}_0^{(N)} + m^2(t)\chi_0^{(N)} = 0. \tag{48}$$

Attractor behavior during $t < t_c$ gives for the solutions

$$\chi_0 \approx e_1 h_1^{(N)}(1 + O(e^{-2\nu(m_1)H(t_c - t_k)})) \tag{49}$$

$$\delta\chi_{nad} \approx c_1 h_1^{(N)}(1 + O(e^{-2\nu(m_1)H(t_c - t_k)})) \tag{50}$$

in the language of the proof of theorem 1. Using the well known "sudden" approximation, one can match these solutions valid for $t < t_c$ to those valid for $t > t_c$:

$$e_1 \begin{pmatrix} h_1^{(N)} \\ \dot{h}_1^{(N)} \end{pmatrix}_{t=t_c} = \begin{pmatrix} H_1^{(N)} & H_2^{(N)} \\ \dot{H}_1^{(N)} & \dot{H}_2^{(N)} \end{pmatrix}_{t=t_c} \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} \tag{51}$$

$$c_1 \begin{pmatrix} h_1^{(N)} \\ \dot{h}_1^{(N)} \end{pmatrix}_{t=t_c} = \begin{pmatrix} H_1^{(N)} & H_2^{(N)} \\ \dot{H}_1^{(N)} & \dot{H}_2^{(N)} \end{pmatrix}_{t=t_c} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \tag{52}$$

where E_i and C_i are independent solution coefficients specifying the χ_0 and $\delta\chi_{nad}$ solutions (respectively) in the time region $t > t_c$ and $H_i^{(N)}$ are independent solutions in the $t > t_c$ time region. Clearly, we have in the region $t > t_c$

$$\frac{\delta\chi_{nad}}{\chi_0} = \frac{c_1}{e_1} \tag{53}$$

which is a constant.

2.3. Theorem 2: Gauge Invariant Isocurvature Spectrum During Radiation Domination

Here is a statement of theorem 2. The radiation dominated period linear spectator isocurvature perturbation spectrum defined by Eq. (11) on superhorizon length scales is given by

$$\Delta_{s_\chi}^2(k) = \frac{k^3}{2\pi^2} \int \frac{d^3k'}{(2\pi)^3} \langle S_\chi(t,\vec{k}) S_\chi(t,\vec{k}')\rangle \tag{54}$$

which is time independent as long as χ interaction is dominated by $V_\chi = m^2\chi^2/2$ and gravity, anisotropic stress effects can be neglected, slow-roll attractor behavior of the $\chi_0(t)$ *during inflation* is relevant (e.g. boundary conditions close to slow-roll are chosen for the homogeneous field), and $m \gg 3H/2$ during the radiation dominated time period when one wishes to evaluate this expression. An important part of the linear spectator requirement is given by Eq. (7). An interesting point of this theorem is that $\Delta_{s_\chi}^2$ is defined in Eq. (11) with $\dot{\chi}_0^2$ in the denominator while Eq. (54) is proportional to χ_0 in the denominator.

Proof

Consider the computation of

$$\Delta_{s_\chi}^2(k) = \frac{k^3}{2\pi^2} \int \frac{d^3k'}{(2\pi)^3} \langle \delta_{s_\chi}(t,\vec{k}) \delta_{s_\chi}(t,\vec{k}') \rangle. \tag{55}$$

during radiation domination when $m \gg 3H/2$. Since $\chi_0(t)$ is coherently oscillating, its energy density consists mostly of non-relativistic energy density. Hence, the definition of

$$\delta\chi_{nad} \equiv \delta\chi^{(N)} - \delta\chi_{ad}^{(N)} \tag{56}$$

gives

$$\delta_{s_\chi} = \frac{\dot{\chi}_0 \partial_t \delta\chi_{nad}|_{\text{background smoothed}} + m^2 \delta\chi_{nad}|_{\text{background smoothed}}}{\langle \dot{\chi}_0^2 \rangle_{\text{time}}} \tag{57}$$

according to Eq. (13) where we also defined the term “background smoothed.”

From the definition of the conserved quantity $S_\chi^{(G)}$ in Eq. (17), we can make the substitution

$$\delta\chi_{nad} = \frac{\chi_0(t)}{2} S_\chi. \tag{58}$$

Hence, the isocurvature classical quantity during radiation domination simplifies to

$$\delta_{s_\chi} = \frac{S_\chi}{2}\left[1 + \frac{m^2 \langle \chi_0^2 \rangle_{\text{time}}}{\langle \dot{\chi}_0^2 \rangle_{\text{time}}}\right] \tag{59}$$

where we have used theorem 1 in keeping S_χ constant, and the time average is given by

$$\begin{aligned} \langle \rho + P \rangle_{\text{time}} &= \langle \dot{\chi}_0^2 \rangle_{\text{time}} && (60) \\ &= m^2 \langle \chi_0^2 \rangle_{\text{time}}. && (61) \end{aligned}$$

This gives

$$\delta_{s_\chi} = S_\chi. \tag{62}$$

We thus conclude that $\Delta_{s_\chi}^2(k)$ is asymptotically time invariant during this radiation dominated time period when the conditions of theorem 1 are valid approximations.

2.4. Theorem 3: Quantum Correlator of Linear Spectator Isocurvature Perturbations

In addition to the conditions of theorem 2, if Bunch-Davies boundary conditions to the inflationary quantization are imposed, $m < 3H(\text{during inflation})/2$, and a slow-roll inflationary phase characterized by the slow-roll function $\varepsilon \ll 1$ defined by $\dot{H} = -\varepsilon H^2$ during inflation occurs, the

spectrum of linear spectator isocurvature perturbations on superhorizon scales during radiation domination is given by

$$\Delta_{s_\chi}^2(k) \approx 4\left(\frac{2^{2\nu-1}|\Gamma(\nu)|^2}{\pi}\right)\left(\frac{H(t_{k_0})}{2\pi\chi_0(t_{k_0})}\right)^2\left(\frac{k}{k_0}\right)^{3-2\nu-2\varepsilon_{k_0}+O(\varepsilon_{k_0})g_0(m/H)} \tag{63}$$

where t_k satisfies $k/a(t_k) = H(t_k)$ during inflation, $g_0(m/H)$ is an order unity function which vanishes when $m/H \to 0$, and

$$\nu \equiv \frac{3}{2}\sqrt{1-\frac{4}{9}\frac{m^2}{H^2(t_{k_0})}} \tag{64}$$

with k_0 being a fiducial wave vector (typically taken on CMB length scales). Here, this expression is only valid when

$$\nu\tilde{N}_k \gg |\ln f| \tag{65}$$

where $\tilde{N}_k \in \{f/\varepsilon_k, H(t_e - t_k)\}$. Here, t_k is the time of the horizon exit of mode k, t_e is the end of inflation, and f is the fractional error tolerance in the computation. Moreover, for ν to be real and also for the correction to $2^\nu|\Gamma(\nu)|^2$ to be within the error tolerance throughout the time period between when the mode $k > k_0$ leaves the horizon and the mode k_0 leaves the horizon, we must have

$$2\varepsilon_{k_0}\frac{m^2}{H^2(t_{k_0})\nu^2}\ln\left(\frac{k}{k_0}\right) < f \quad \text{for } k > k_0. \tag{66}$$

The bounds on ν sets the largest valid isocurvature spectral index of

$$n = 4 - 2\nu \tag{67}$$

coming from Eq. (63). Finally, since $\chi_0(t_k)$ was approximated by expanding about t_{k_0}, we must have

$$k_0 \exp\left(-\frac{f}{\varepsilon_{k_0}}\right) < k < k_0 \exp\left(\frac{f}{\varepsilon_{k_0}}\right). \tag{68}$$

The fractional error $O(\mathscr{E})$ in Eq. (63) is approximated to be the terms that are dropped in making this statement:

$$\mathscr{E} = \exp\left[-2\nu(m_1)\tilde{N}_k\right] + \frac{\delta\rho_\chi^{(N)}}{\delta\rho_{\text{dominant}}^{(N)}} + \frac{m^2}{3\sqrt{2\varepsilon}H^2}\frac{|\chi_0(t_k)|}{M_p} + \frac{|\chi_0(t_k)|}{M_p} + \varepsilon_{k_0} + f. \tag{69}$$

Because we are neglecting $O(\varepsilon_{k_0})$ contributions, this result is applicable to very blue spectra in which $3-2\nu \gg \varepsilon_{k_0}$, and because $g_0(0) = 0$, this result is applicable to $3-2\nu = 0$ case.

Before moving onto the proof, let us make some comments. One immediate implication of this theorem is that large isocurvature spectra with blue spectral indices are easy to generate for masses of order H. For example, the supergravity η-problem for the inflaton can be translated to the isocurvature sector to generically expect $m^2/H^2(t_{k_0}) \sim O(1)$ during inflation. On the other hand, this does not mean that such spectra are measurable since any observable must be multiplied by the background energy density of χ, which tends to dilute away when m/H is not small (as noted by [98]). Still, there are examples [1] of supergravity models generating measurable, strongly blue spectrum. This will be discussed in depth in Sec. 3.1. In many applications, $\sqrt{\varepsilon} \ll 1$ which means that Eq. (69) imposes a constraint on $|\chi_0(t_k)|$ for the longest wavelengths leaving the horizon.

proof

To make an inflationary prediction based on theorem 2, we need to match the classical gauge invariant quantity

$$\Delta^2_{s_\chi}(k) = \frac{k^3}{2\pi^2}\int \frac{d^3k'}{(2\pi)^3}\frac{4}{\chi_0^2}\langle\delta\chi_{nad}(t,\vec{k})\delta\chi_{nad}(t,\vec{k}')\rangle \tag{70}$$

to a quantum correlator computation. A technical difficulty lies in the fact that $\delta\chi_{nad}(t,\vec{k})$ is not the field that is being matched to the quantized field (recall it is only a particular part of a classical solution whose initial condition statistics are captured by the correlator of Eq. (70)). One way to quantize is to quantize the modes of the linearized equation of motion for $\delta\chi^{(G)}(t,\vec{k})$ in a particular gauge G. It is important to note that we only need the correlator value computed from field quantization during the few efolds of the horizon exit since by that time the correlator evolves as a classical statistical object and $\Delta^2_{s_\chi}(k)$ is frozen according to theorem 1. If this were not true, we need to understand the full time evolution of the quantum correlator which is a messy task. Below, we will quantize in the spatially flat gauge ($G = sf$) instead of the Newtonian gauge ($G = N$) because the gravitational potential does not appear in the coupled oscillator equations. After quantizing, one can compute the correlator of Eq. (70) by computing $\langle\delta\chi^{(sf)}_{\vec{k}}\delta\chi^{(sf)}_{\vec{p}}\rangle$ and $\langle\delta\chi^{(sf)}_{\vec{k}}\zeta_{\vec{p}}\rangle$.

Let us now consider the details obtaining Eq. (70) by matching to the quantum computation during a few efolds of horizon exit. Since the relationship between the spatially flat gauge isocurvature field and the Newtonian gauge field is

$$\delta\chi^{(sf)}(t,\vec{x}) = \delta\chi^{(N)}(t,\vec{x}) - \frac{\Phi^{(N)}}{H}\dot{\chi}_0(t), \tag{71}$$

we have

$$\left\langle \frac{\delta\chi^{(sf)}_{\vec{k}}}{\chi_0}\frac{\delta\chi^{(sf)}_{\vec{p}}}{\chi_0}\right\rangle = \left\langle \frac{\delta\chi^{(N)}_{\vec{k}}}{\chi_0}\frac{\delta\chi^{(N)}_{\vec{p}}}{\chi_0}\right\rangle + \frac{1}{\chi_0^2}\left(\frac{\dot{\chi}_0}{H}\right)^2\left\langle \Phi^{(N)}_{\vec{k}}\Phi^{(N)}_{\vec{p}}\right\rangle$$
$$-\left(\frac{\dot{\chi}_0}{H}\right)\frac{1}{\chi_0}2\Re\left\langle \frac{\delta\chi^{(N)}_{\vec{k}}}{\chi_0}\Phi^{(N)}_{\vec{p}}\right\rangle. \tag{72}$$

Because of the linear spectator property, $\Phi^{(N)}_{\vec{p}}$ during inflation is given by the adiabatic mode. We parameterize $\delta\chi^{(N)}_{\vec{p}}$ using the nonadiabatic mode $\delta\chi^{(N)}_{nad}(t,\vec{p})$:

$$\delta\chi^{(N)}_{\vec{p}} \approx \delta\chi^{(N)}_{ad}(t,\vec{p})+\delta\chi^{(N)}_{nad}(t,\vec{p}) = -\zeta_{\vec{p}}\frac{\dot{\chi}}{a}\int dta+\delta\chi^{(N)}_{nad}(t,\vec{p}) \tag{73}$$
$$\Phi^{(N)}_{\vec{p}} \approx \zeta_{\vec{p}}\left[1-\frac{H}{a}\int dta\right]. \tag{74}$$

We find

$$\left\langle \frac{\delta\chi^{(sf)}_{\vec{k}}}{\chi_0}\frac{\delta\chi^{(sf)}_{\vec{p}}}{\chi_0}\right\rangle \approx \left\langle \frac{\delta\chi^{(N)}_{nad}(t,\vec{k})}{\chi_0}\frac{\delta\chi^{(N)}_{nad}(t,\vec{p})}{\chi_0}\right\rangle + \langle\zeta_{\vec{k}}\zeta_{\vec{p}}\rangle\left(\frac{\dot{\chi}_0}{H\chi_0}\right)^2$$
$$-2\Re\left\langle \frac{\delta\chi^{(N)}_{nad}(t,\vec{k})}{\chi_0}\zeta_{\vec{p}}\right\rangle\left(\frac{\dot{\chi}_0}{H\chi_0}\right). \tag{75}$$

To extract $\left\langle \delta\chi^{(N)}_{nad}(t,\vec{k})\delta\chi^{(N)}_{nad}(t,\vec{p})\right\rangle$ using the spatially flat gauge correlators, we also need an expression for $\left\langle \delta\chi^{(N)}_{nad}(t,\vec{k})\zeta_{\vec{p}}\right\rangle$ in terms of $\left\langle \delta\chi^{(sf)}_{\vec{k}}\zeta_{\vec{p}}\right\rangle$. Following a similar procedure as before, this is given by

$$\left\langle \frac{\delta\chi^{(sf)}_{\vec{k}}}{\chi_0}\zeta_{\vec{p}}\right\rangle = \frac{1}{\chi_0}\left\langle \delta\chi^{(N)}_{nad}(t,\vec{k})\zeta_{\vec{p}}\right\rangle - \langle\zeta_{\vec{k}}\zeta_{\vec{p}}\rangle\frac{\dot{\chi}_0}{H\chi_0}. \tag{76}$$

Hence, we have arrived at the desired expression

$$4\left\langle \frac{\delta\chi^{(N)}_{nad}(t,\vec{k})}{\chi_0}\frac{\delta\chi^{(N)}_{nad}(t,\vec{p})}{\chi_0}\right\rangle \approx 4\left\langle \left(\frac{\delta\chi^{(sf)}_{\vec{k}}}{\chi_0}+\zeta_{\vec{k}}\left(\frac{\dot{\chi}_0}{H\chi_0}\right)\right)\right.$$
$$\left.\left(\frac{\delta\chi^{(sf)}_{\vec{p}}}{\chi_0}+\zeta_{\vec{p}}\left(\frac{\dot{\chi}_0}{H\chi_0}\right)\right)\right\rangle \tag{77}$$

where the left hand side is interpreted as a correlator of stochastic classical fluctuations and the right hand side is computed at the quantized level. Note that this expression indicates what is conserved is effectively the naively expected quantity $\delta\chi^{(sf)}/\chi_0$ with the gravitational potential effect subtracted. One may naively think that the second term of Eq. (77) contributes significantly when $m/H \sim O(1)$. However, we will show below how this term is exactly canceled out.

Let us now compute the correlators explicitly in the leading slow-roll approximation. We recall that in the spatially flat gauge, the mode equations are homogeneous with off-diagonal mixing

$$\left(\partial_t^2 + 3H\partial_t + \mathbf{M}^2\right)\begin{pmatrix}\delta\varphi^{(sf)}\\ \delta\chi^{(sf)}\end{pmatrix} = 0 \tag{78}$$

which allows one to use adiabatic approximation to quantize the system since the mass matrix $\mathbf{M}^2$ varies sufficiently slowly in time during slow-roll period for approximate diagonalization. Making a time independent rotation to the approximately diagonal basis, we find the approximately diagonal modes $h_{\varphi,\chi}^{(sf)}$ satisfy

$$\left(\partial_t^2 + 3H\partial_t + \mathbf{U}^\dagger\mathbf{M}^2\mathbf{U}\right)\begin{pmatrix}\delta\varphi_d^{(sf)}\\ \delta\chi_d^{(sf)}\end{pmatrix} = 0 \tag{79}$$

where

$$\begin{aligned} M_{11}^2 &= V''_\varphi + \frac{3}{M_p^2}\dot\varphi_0^2 - \frac{1}{2}\frac{\dot\varphi_0^4}{M_p^4H^2} + 2\frac{V'_\varphi\dot\varphi_0}{M_p^2H} - \frac{\dot\varphi_0^2\dot\chi_0^2}{2M_p^4H^2} && (80)\\ &\approx (3\eta_V - 6\varepsilon)H^2 && (81)\end{aligned}$$

$$M_{12}^2 = M_{21}^2 = -\frac{1}{2}\left[-6\frac{\dot\varphi_0\dot\chi_0}{M_p^2} + \frac{\dot\varphi_0^3\dot\chi_0}{M_p^4H^2} + \frac{\dot\varphi_0\dot\chi_0^3}{M_p^4H^2} - 2\frac{\dot\varphi_0 V'_\chi}{M_p^2H} - 2\frac{\dot\chi_0 V'_\varphi}{M_p^2H}\right] \tag{82}$$

$$M_{22}^2 = V''_\chi + \frac{3}{M_p^2}\dot\chi_0^2 - \frac{1}{2}\frac{\dot\chi_0^4}{M_p^4H^2} + 2\frac{V'_\chi\dot\chi_0}{M_p^2H} - \frac{\dot\varphi_0^2\dot\chi_0^2}{2M_p^4H^2} \tag{83}$$

and $\mathbf{U}^\dagger\mathbf{M}^2\mathbf{U}$ is a diagonal matrix. Let's define

$$\kappa \equiv -\frac{M_{12}^2}{M_{22}^2}|_{t=0} \approx \frac{-\frac{\dot\varphi_0 V'_\chi(\chi)}{M_p^2H}}{m^2[1+O\left(\frac{\chi_0^2}{M_P^2}\right)]} = \frac{\sqrt{2\varepsilon}\chi_0}{M_p}\mathrm{sgn}(\dot\varphi_0) \tag{84}$$

$$L \equiv -\frac{M_{22}^2}{M_{11}^2}|_{t=0} \approx -\frac{m^2[1+O\left(\frac{\chi_0^2}{M_P^2}\right)]}{(3\eta_V - 6\varepsilon)H^2} \tag{85}$$

where $\kappa \ll 1$ because of Eq. (9). As far as L is concerned, consider two cases. One case will be $|L| \gg 1$ and the other case will be $|L| \ll 1$. The former is the more interesting case, since the other case is the one which people have phenomenologically considered most commonly and has been well established.

Case with $|L| \gg 1$

$$\mathbf{U}_{11} = 1 - \frac{\kappa^2\lambda_M^2}{2} + \frac{\kappa^2\lambda_M^3}{L} + \lambda_M^4\left(\frac{11\kappa^4}{8} - \frac{3\kappa^2}{2L^2}\right) + ... \tag{86}$$

$$\mathbf{U}_{22} = 1 - \frac{\kappa^2\lambda_M^2}{2} + \frac{\kappa^2\lambda_M^3}{L} + \lambda_M^4\left(\frac{11\kappa^4}{8} - \frac{3\kappa^2}{2L^2}\right) + ... \tag{87}$$

$$\mathbf{U}_{12} = -\kappa\lambda_M + \frac{\kappa\lambda_M^2}{L} + \lambda_M^3\left(\frac{3\kappa^3}{2} - \frac{\kappa}{L^2}\right) + \lambda_M^4\left(\frac{\kappa}{L^3} - \frac{9\kappa^3}{2L}\right) + ... \tag{88}$$

$$\mathbf{U}_{21} = \kappa\lambda_M - \frac{\kappa\lambda_M^2}{L} - \lambda_M^3\left(\frac{3\kappa^3}{2} - \frac{\kappa}{L^2}\right) - \lambda_M^4\left(\frac{\kappa}{L^3} - \frac{9\kappa^3}{2L}\right) + ... \tag{89}$$

with formal perturbation power λ_M assignment as $M_{11}^2(0) = O(\lambda_M)$, $M_{12}^2(0) = O(\lambda_M)$, $M_{22}^2(0) = O(1)$ and $t = 0$ is defined to be initial time here when the modes are deep within the horizon (i.e. when $k/a(0) \gg H$). Note that this diagonalization differs in a numerically insignificant manner from that used in [99] where the diagonalization is carried out the horizon exit time (See Eq. (8) in [99]).[4] Hence, because

$$\frac{\chi_0}{M_p} \ll 1, \tag{90}$$

the off-diagonal element of the mixing matrix $\mathbf{U}$ is certainly generically negligible at the initial time.

Based on the smallness of mixing in $\mathbf{U}$, one might naively think that the mixing continues to be negligible for the far superhorizon evolution of $\delta\chi^{(sf)}$ if $\chi_0/M_p \ll 1$ since

$$\frac{d^2\delta\chi^{(sf)}}{dt^2} + 3H\delta\chi^{(sf)} + \left(\frac{k^2}{a^2} + M_{22}^2\right)\delta\chi^{(sf)} = O(\chi_0/M_p)\delta\varphi^{(sf)} \tag{91}$$

which may naively allow us to neglect the M_{21}^2 dependent right hand side in the limit $\chi_0/M_p \to 0$. However, this is not quite right. Since $\kappa \ll 1$ and $L \gg 1$ which implies

$$\frac{3}{2}H^2 > M_{22}^2 \gg M_{11}^2, M_{12}^2, \tag{92}$$

[4] Since the vacuum physically corresponds to zero on-shell particle states when the modes are approximately Minkowskian, diagonalization of the mass matrix when the modes are deep within the horizon is physically more faithful. On the other hand, as long as the mass matrix is sufficiently slow in its time dependence, the distinction is not numerically important. One might also argue that a time dependent diagonalization in the spirit of WKB approximation is even more appropriate when the modes are deep within the horizon because as shown in [100], the time dependent rotation always generates mixing at any time $\dot{\chi}_0/\dot{\varphi}_0 \neq 0$. However, as noticed in [99], the ratio of the off-diagonal mass squared and the diagonal mass squared is proportional to slow-roll parameters whose time variation is suppressed by higher order in slow-roll. In that sense, neglecting the *time dependence* of the mixing matrix is justified.

the field $\delta\varphi^{(sf)}$ will *eventually* grow relative to $\delta\chi^{(sf)}$: to see this, use the trial superhorizon positive frequency mode solution $e^{-i\omega t}$ and consider the resulting dispersion relationships in the constant H limit. In contrast, in the decoupling limit, $\delta\varphi^{(sf)}$ will evolve independently of $\delta\chi^{(sf)}$.[5]Because of these two features, the mode $\delta\chi^{(sf)}$ obtains an effectively inhomogeneous (i.e. sourced) contribution proportional to $\delta\varphi^{(sf)}$:

$$\delta\chi^{(sf)}_{\text{particular}}(t) \approx \frac{\dot{\chi}_0(t)}{\dot{\varphi}_0(t)}\delta\varphi^{(sf)}(t) \tag{93}$$

where we have used slow-roll equations of motion but have *not* used the superhorizon approximation.[6] The fact that this is a valid solution in the subhorizon region is explicitly demonstrated in Appendix A. On top of this, one can add a homogeneous solution

$$\delta\chi^{(sf)}(t) \approx Ch^{(sf)}_{\chi}(t) + \frac{\dot{\chi}_0(t)}{\dot{\varphi}_0(t)}\delta\varphi^{(sf)}(t) \tag{94}$$

where $h^{(sf)}_{\chi}$ is a free oscillator field satisfying

$$\frac{d^2 h^{(sf)}_{\chi}}{dt^2} + 3Hh^{(sf)}_{\chi} + \left(\frac{k^2}{a^2} + M_{22}^2\right) h^{(sf)}_{\chi} = 0 \tag{95}$$

with the normalization

$$\langle h^{(sf)}_{\chi}(t,\vec{k}) h^{(sf)}_{\chi}(t,\vec{p})\rangle \underset{k/a\to\infty}{\sim} \frac{1}{2ka^2}(2\pi)^3\delta^{(3)}(\vec{k}+\vec{p}) \tag{96}$$

and C is a coefficient determined by Bunch-Davies boundary conditions (note the basis h defined here already satisfies the Bunch-Davies boundary conditions). Note that if H and M_{22} are constants, $h^{(sf)}_{\chi}$ is composed of Hankel functions.

Another way to justify the decomposition into the particular and homogeneous solution here is that we have to *keep* the second term of Eq. (94) which goes as

$$\frac{\dot{\chi}_0(t)}{\dot{\varphi}_0(t)}\delta\varphi^{(sf)}(t) \approx \frac{-m^2}{3\sqrt{2\varepsilon}H^2 \operatorname{sgn}\dot{\varphi}_0}\frac{\chi_0}{M_p}\delta\varphi^{(sf)}(t) \tag{97}$$

even in the decoupling limit of $\chi_0/M_p \ll 1$ because of the unsuppression due to the relative growing nature of $\delta\varphi^{(sf)}$ relative to $\delta\chi^{(sf)}$. This contribution is small on subhorizon scales but grows

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[5] Neglecting $\delta\chi^{(sf)}$ influence on $\delta\varphi^{(sf)}$ is certainly valid since the mixing is small and $\delta\varphi^{(sf)}$ grows relative to $\delta\chi^{(sf)}$ when Eq. (92) is satisfied as one can see by considering the positive frequency mode equations.

[6] In spatially flat gauge, this particular solution is related to the gauge invariant perturbations through $\zeta \approx -H\delta\varphi^{(sf)}/\dot{\varphi}_0$.

after the modes leave the horizon. This is a type of secular effect in which long time behavior unsuppresses the naively suppressed contribution.

Let's now estimate C in Eq. (94). By definition

$$\delta\chi^{(sf)} = \mathbf{U}_{21}\delta\varphi_d^{(sf)} + \mathbf{U}_{22}\delta\chi_d^{(sf)} \tag{98}$$

where $\delta\varphi_d^{(sf)}$ and $\delta\chi_d^{(sf)}$ are decoupled at the initial time $t=0$, with mass matrix eigenvalues m_1^2 and m_2^2 respectively (see Eq. (79)). To determine C, we want to match Eqs. (94) and (98). Note that the decoupling used in Eq. (94) already drops $O(\chi_0/M_p)$ contributions **except** for those that can eventually be unsuppressed by the relative growth of $\delta\varphi^{(sf)}$ compared to $\delta\chi^{(sf)}$ after the modes leave the horizon.[7] In the subhorizon region, there is no enhancement due to relative growth of $\delta\varphi^{(sf)}$ to $\delta\chi^{(sf)}$. Since we will be matching in the subhorizon region, we can drop the $\mathbf{U}_{12}$ and $\mathbf{U}_{21}$ terms when matching:

$$\delta\chi^{(sf)} \;=\; Ch_\chi^{(sf)}(t) + \frac{\dot{\chi}_0(t)}{\dot{\varphi}_0(t)}\delta\varphi^{(sf)}(t) \approx \delta\chi_d^{(sf)}(t) \tag{99}$$

$$\delta\phi^{(sf)} \;\approx\; \delta\phi_d^{(sf)}(t) \tag{100}$$

At earlier times when the modes are subhorizon, from the normalization (96) we have

$$\left\langle \delta\chi_d^{(sf)}(t,\vec{k})\delta\chi_d^{(sf)}(t,\vec{p})\right\rangle \underset{k/a\to\infty}{\sim} \left\langle h_\chi^{(sf)}(t,\vec{k})h_\chi^{(sf)}(t,\vec{p})\right\rangle \tag{101}$$

if

$$m_2^2 \approx M_{22}^2 \tag{102}$$

consistently with Eq. (92). Then Eq. (99) gives

$$\begin{aligned}|C|^2\left\langle h_\chi^{(sf)}(t,\vec{k})h_\chi^{(sf)}(t,\vec{p})\right\rangle \;\approx\; &\left\langle \delta\chi_d^{(sf)}(t,\vec{k})\delta\chi_d^{(sf)}(t,\vec{p})\right\rangle \\ &+\left(\frac{\dot{\chi}_0(t)}{\dot{\varphi}_0(t)}\right)^2\left\langle \delta\phi_d^{(sf)}(t,\vec{k})\delta\phi_d^{(sf)}(t,\vec{p})\right\rangle .\end{aligned} \tag{103}$$

Assuming $\left\langle \delta\chi_d^{(sf)}(t,\vec{k})\delta\chi_d^{(sf)}(t,\vec{p})\right\rangle \sim \left\langle \delta\phi_d^{(sf)}(t,\vec{k})\delta\phi_d^{(sf)}(t,\vec{p})\right\rangle$ in the $k/a\to\infty$ limit, we can conclude that

$$C = 1 + O\left(\frac{m^2}{3\sqrt{2\varepsilon}H^2}\frac{\chi_0}{M_p}\right). \tag{104}$$

[7] Although $\mathbf{U}_{21}\delta\varphi_d^{(sf)}$ in Eq. (98) is also unsuppressed by $\delta\varphi_d^{(sf)}$ in the superhorizon region, the coefficient $\mathbf{U}_{21} = O(\sqrt{\varepsilon}\kappa)$ has an extra power of slow roll parameter ε when compared to Eq. (97). We do not really need this fact for the demonstration here since we are matching in the subhorizon region. It is being mentioned to note that the second term of Eq. (94) is not coming from $\mathbf{U}_{21}$ which is something that is being evaluated only at the initial time.

Eq. (99) thus can be written as

$$\frac{\delta\chi_{\vec{k}}^{(sf)}}{\chi_0}+\zeta_{\vec{k}}\left(\frac{\dot{\chi}_0}{H\chi_0}\right)=\frac{h_{\chi}^{(sf)}(t,\vec{k})}{\chi_0} \tag{105}$$

where we used that in spatially flat gauge

$$\zeta=-H\delta\varphi^{(sf)}/\dot{\varphi}_0. \tag{106}$$

Hence, approximating the Bunch-Davies state correlator near horizon exit time as

$$\langle h_{\chi}^{(sf)}(t,\vec{k})h_{\chi}^{(sf)}(t,\vec{p})\rangle=\frac{(2\pi)^3}{a^3}\frac{\pi}{4}\frac{1}{H}\left|H_{\nu}^{(1)}(\frac{k}{aH})\right|^2\delta^{(3)}(\vec{k}+\vec{p}) \tag{107}$$

where

$$\nu=\frac{3}{2}\sqrt{1-\frac{4}{9}\frac{m^2}{H^2}}, \tag{108}$$

we expand this in the limit $k/(aH)\to 0$ in the usual manner to arrive at

$$\left\langle\frac{\delta\chi_{nad}^{(N)}(t,\vec{k})}{\chi_0}\frac{\delta\chi_{nad}^{(N)}(t,\vec{p})}{\chi_0}\right\rangle\approx(2\pi)^3\delta^{(3)}(\vec{k}+\vec{p})\left\{\frac{1}{a^3\chi_0^2}2^{2(\nu-1)}|\Gamma(\nu)|^2\left(\frac{k}{aH}\right)^{-2\nu}\frac{1}{H\pi}\right\} \tag{109}$$

which is only valid when

$$\left|\nu\ln\left(\frac{k}{aH}\right)\right|\gg|\ln f| \tag{110}$$

where f is the fractional accuracy desired and H is approximately constant. Even Eq. (110) is about justifying keeping the leading term in the Hankel function expansion, we will soon discuss that this also corresponds to the attractor behavior needed for the validity of theorem 1 that will be used here.

Eq. (109) is not manifestly frozen since for example the first term seems to dilute as a^3. However, just as we stated in theorem 1 (which is more general for classical solutions than this computation of a few efolds during inflation), this expression does freeze. To see this, during the few efolds period of horizon exit during inflation, we can solve

$$\ddot{\chi}_0+3H\dot{\chi}_0+m^2\chi_0=0 \tag{111}$$

to obtain the attractor solution

$$\chi_0(t)\approx\chi_0(t_k)\left(\frac{a(t)}{a(t_k)}\right)^{-\frac{3}{2}+\nu+O(\varepsilon)g_0(m/H)} \tag{112}$$

where $k/a(t_k) = H(t_k)$, $g_0(x)$ is a function which vanishes in the limit $m/H \to 0$, and the approximation drops the decaying solution. Hence, one sees that all the $a(t)$ dependencies cancel in Eq. (109) and are replaced by $a(t_k)$.

Let us now consider matching the quantum computation to the classical variable of theorem 1. With the usual Bunch-Davies normalizations, $\delta\chi_{nad}$ on superhorizon scales is determined by purely imaginary mode functions: i.e. the time dependence of mode functions are determined by by $H_\nu^{(1)}(k/(aH))$ where

$$H_\nu^{(1)}(x) = J_\nu^{(1)}(x) + iY_\nu^{(1)}(x) \tag{113}$$

where the $Y_\nu^{(1)}(x \to 0) \to \infty$. Thus $\delta\chi_{nad}$ commute like a classical random variables once $Y_\nu^{(1)}$ dominates over $J_\nu^{(1)}$. The time scale on which $Y_\nu^{(1)}$ dominates over $J_\nu^{(1)}$ is

$$\tau = \frac{1}{2\nu H}. \tag{114}$$

This is coincidentally the same time scale as the classical attractor behavior discussed in theorem 1 and approximately the same time scale after which Eq. (110) becomes satisfied. Once

$$\exp[-(t - t_k)/\tau] \ll f \tag{115}$$

condition is satisfied, one can match the correlator of classical random variable S_χ to the quantum correlator on superhorizon scales using Eq. (77):

$$\langle S_\chi S_\chi \rangle = 4(2\pi)^3 \delta^{(3)}(\vec{k} + \vec{p}) \left\{ \frac{1}{a^3 \chi_0^2} 2^{2(\nu-1)} |\Gamma(\nu)|^2 \left(\frac{k}{aH} \right)^{-2\nu} \frac{1}{H\pi} \right\}. \tag{116}$$

In other words, to use theorems 1 and 2 to connect late time isocurvature to the quantum correlator, we must satisfy Eq. (110).

Finally, it is instructive to convert Eq. (116) into the form of Eq. (54). We start by using Eq. (112) to recast Eq. (116) as

$$\Delta_{s_\chi}^2(k) = 4 \left[\frac{2^{2\nu} H^2 |\Gamma(\nu)|^2}{\chi_0^2(t_k)(2\pi)^3} \right] \tag{117}$$

where one must keep in mind that despite its appearance, we have already evaluated this at a time when $a(t) \gg a(t_k)$ and we have already assumed Eq. (110) is satisfied (we will find the constraint imposed by the assumption below). Choose a time t_{k_0} corresponding to a fiducial mode horizon exit time to write

$$H(t_k) = H(t_{k_0}) \left(\frac{k}{k_0} \right)^{-\varepsilon} \tag{118}$$

$$\frac{a(t_{k_0})}{a(t_k)} = \left(\frac{k}{k_0}\right)^{-\varepsilon-1} \tag{119}$$

implying

$$\Delta^2_{s_\chi}(k) = 4\left[\frac{2^{2\nu}H^2(t_{k_0})|\Gamma(\nu)|^2}{\chi_0^2(t_{k_0})(2\pi)^3}\right]\left(\frac{k}{k_0}\right)^{(3-2\nu)-2\varepsilon+O(\varepsilon)g_0(m/H)}. \tag{120}$$

Note also that we are keeping the $\ln(k/k_0)$ enhanced ε dependence while dropping other non-enhanced ε dependences. Since we only specify that $g_0(m/H)$ vanishes when $m = 0$, the -2ε power is numerically meaningful in the current estimate only when $m = 0$, which is really not about the blue spectra. Nonetheless, we keep it here to connect this spectra to the massless axion spectra.

Let us now find the parametric region where one expects the attractor to be reached consistently with Eq. (110). An obstacle to satisfying Eq. (110) in the context of Eq. (117) is that either inflation ends too early or the assumption that H is constant during the realization of Eq. (115) is invalid. The assumption of constant H breaks down on a time scale Δt satisfying

$$|\dot{H}|\Delta t = fH \tag{121}$$

where f is the accuracy desired. During slow-roll, we have

$$\Delta t = \frac{f}{\varepsilon H}. \tag{122}$$

Hence, Eq. (110) becomes

$$\nu\tilde{N}_k \gg |\ln f| \tag{123}$$

where $\tilde{N}_k \in \{f/\varepsilon_k, H(t_e - t_k)\}$ where t_k is the time of the horizon exit of mode k and t_e is the end of inflation.

Note also even if $|\dot{H}|\Delta t/H$ is within error tolerance f, it may still be bigger than what is need to keep ν real. Expressed in terms of ν with H evaluated at the fiducial k_0 horizon crossing, we need

$$1 - \frac{4}{9}\frac{m^2}{(H(t_{k_0}) - \varepsilon_{k_0}H^2(t_{k_0})(t_k - t_{k_0}))^2} > 0 \tag{124}$$

which translates to

$$2\varepsilon_{k_0}\frac{m^2}{H^2(t_{k_0})\nu^2}\ln(k/k_0) < 1 \quad \text{for } k > k_0. \tag{125}$$

Furthermore, the prefactor terms $2^\nu|\Gamma(\nu)|^2$ and $\chi_0(t_k)$ were approximated by expanding about t_{k_0} assuming a constant H. We require the correction to $2^\nu|\Gamma(\nu)|^2$ to be within the error tolerance.

This means that in considering the large enhancement situation $2^{\nu}|\Gamma(\nu)|^2 \approx 1/\nu^2$ near $\nu = 0$, we impose

$$2^{\nu_k}|\Gamma(\nu_k)|^2 \approx \frac{1}{\nu^2}\left(1 + 2\varepsilon_{k_0}\frac{m^2}{H^2(t_{k_0})\nu^2}\ln(k/k_0)\right) < \frac{1}{\nu^2}(1+f), \tag{126}$$

that gives Eq. (66). Finally, Eq. (122) restricts k through the horizon crossing considerations to be in the range

$$k_0 \exp\left(-\frac{f}{\varepsilon_{k_0}}\right) < k < k_0 \exp\left(\frac{f}{\varepsilon_{k_0}}\right). \tag{127}$$

Case with $|L| \ll 1$ In this case, we have

$$L \equiv -\frac{M_{22}^2}{M_{11}^2}|_{t=0} \approx -\frac{m^2[1 + O\left(\frac{\chi^2}{M_P^2}\right)]}{(3\eta_V - 6\varepsilon)H^2} \ll 1. \tag{128}$$

Because we still have $\chi_0/M_p \ll 1$ and

$$M_{21}^2 \approx \frac{m^2\chi_0\sqrt{2\varepsilon}}{M_p}\mathrm{sgn}\dot{\varphi}_0, \tag{129}$$

we have the hierarchy

$$M_{12}^2 = M_{21}^2 \ll M_{22}^2 \ll M_{11}^2. \tag{130}$$

Hence, the off-diagonal element of the mixing matrix $\mathbf{U}$ is still suppressed. (In fact, it is now suppressed relative to all of the matrix elements.) Furthermore, again because of the mass hierarchy in Eq. (130), $\delta\varphi^{(sf)}$ does not grow relative to $\delta\chi^{(sf)}$ unlike the situation explained just after Eq. (91). Hence, we can in this case completely ignore the mixing.

Finally, we see the term shifting the $\delta\chi^{(sf)}$ field in Eq. (77) is negligible to leading slow-roll order accuracy: i.e.

$$\zeta_{\vec{k}}\left(\frac{\dot{\chi}_0}{H\chi}\right) \approx \zeta_{\vec{k}}\left(\frac{-m^2}{3H^2}\right) \ll (3\eta_V - 6\varepsilon)\zeta_k. \tag{131}$$

Hence, we again conclude that the isocurvature spectrum is given by Eq. (117).

Just as for theorem 1, a trivial corollary of this theorem is to discuss the situation when the constant m is replaced by $m(t)$ which is constant during a finite time interval during inflation and makes a transition to another value during inflation.

corollary 2 In the context of theorem 3 and just as in corollary 1, suppose m is not a constant but makes a transition to another value during inflation:

$$m^2(t) = \begin{cases} m_1 & t < t_c \\ m_2 & t > t_c \end{cases} \tag{132}$$

where the transition time region near $t = t_c$ is assumed to be much smaller in time than m_1^{-1} and H^{-1}. Suppose sufficient time has passed during the $t < t_c$ period to be in the attractor approximation just as in corollary 1: i.e.

$$\tilde{N}(t_c, t_k)\nu(m_1) > |\ln f| \tag{133}$$

where $\tilde{N}(t_c, t_k) \in \{f/\varepsilon_k, (t_c - t_k)H\}$,

$$\nu(m_1) = \frac{3}{2}\sqrt{1 - \frac{4}{9}\frac{m_1^2}{H^2}}, \tag{134}$$

and $f < 1$ is the error tolerance in the computation. The spectrum is still given by Eq. (63) with $\nu \to \nu(m_1)$ for modes k in the range

$$k_{\min} < k < k_{\max} \tag{135}$$

where $k_{\max}$ is the smallest k that among $\{k$ that saturates the inequality of Eq. (133), $k_0 \exp(f/\varepsilon_{k_0})$, $k_0 \exp\left[\frac{\nu^2}{2\varepsilon_{k_0}} \frac{H^2(t_{k_0})}{m^2}\right]\}$ and

$$k_{\min} = \max\left\{k_0 \exp\left(-\frac{f}{\varepsilon_{k_0}}\right), a(t_b)H(t_b)\right\} \tag{136}$$

where t_b is the beginning of inflation. The fractional error $O(\mathscr{E})$ in Eq. (63) receives contributions from

$$\mathscr{E} = \exp\left[-2\nu(m_1)(t_c - t_k)H\right] + \frac{\delta\rho_\chi^{(N)}}{\delta\rho_{\text{dominant}}^{(N)}} + \frac{m^2}{3\sqrt{2\varepsilon}H^2}\frac{|\chi_0(t_k)|}{M_p} + \frac{|\chi_0(t_k)|}{M_p} + \varepsilon_{k_0} + f. \tag{137}$$

The error contribution proportional to $1/\sqrt{\varepsilon}$ in Eq. (137) can be rewritten as abound on $\chi_0(t_{k_0})/M_p$ if we require it to be less than f and require $k_{H_0} \approx a_0 H_0 \pi/2$[8] mode to be the quantizable within the current analytic treatment:

$$\frac{2}{3}\pi\left(\frac{k_{H_0}}{k_0}\right)^{-\frac{3}{2}+\nu+O(\varepsilon)g_0(m/H)} \frac{m^2}{H^2}\sqrt{\Delta_\zeta^2(k_{H_0})}\frac{M_p}{H}\frac{1}{f} < \frac{M_p}{|\chi_0(t_{k_0})|}. \tag{138}$$

This and Eq. (16) give

$$\frac{\Delta_s^2(k_{H_0})}{\Delta_\zeta^2(k_{H_0})} = \omega_\chi^2 \frac{\Delta_{s_\chi}^2(k_{H_0})}{\Delta_\zeta^2(k_{H_0})} > \omega_\chi^2 \frac{2^{2\nu-1}|\Gamma(\nu)|^2}{\pi}\left[\frac{2}{3f}\frac{m^2}{H^2}\right]^2. \tag{139}$$

[8] k_{H_0} is the wave number corresponding to the observable universe today that in a comoving box with a length of $L = 4/a_0 H_0$.

Since we know that the left hand side is constrained to a few percent level (at least for the scale invariant spectra), phenomenology requires that the dark matter fraction in χ be very small for a very blue spectra: i.e. $\omega_\chi \ll 1$ for $m/H \sim O(1)$.

proof

For those modes which can have Bunch-Davies boundary conditions at early time and exits the horizon long before $t = t_c$ will lead to a spectrum where Eqs. (109) and (110) are valid with $\nu(m) \to \nu(m_1)$:

$$\left\langle \frac{\delta\chi_{nad}^{(N)}(t,\vec{k})}{\chi_0} \frac{\delta\chi_{nad}^{(N)}(t,\vec{p})}{\chi_0} \right\rangle \approx (2\pi)^3 \delta^{(3)}(\vec{k}+\vec{p}) \left\{ \frac{2^{2(\nu(m_1)-1)}}{a^3\chi_0^2} \frac{|\Gamma(\nu(m_1))|^2}{H\pi} \left(\frac{k}{aH}\right)^{-2\nu(m_1)} \right\} \tag{140}$$

$$\nu(x) \equiv \frac{3}{2}\sqrt{1 - \frac{4}{9}\frac{x^2}{H^2}}. \tag{141}$$

If the dominance of $Y_{\nu(m_1)}(x)$ over $J_{\nu(m_1)}(x)$ occurs as

$$\exp[-(t-t_k)2\nu(m_1)H] \ll f, \tag{142}$$

with $t < t_c$, classical behavior is heuristically justified, and one can match the quantum computation to the classical solution. Since the constant H approximation result of Eq. (140) requires $t < t_k + f/(\varepsilon_k H)$ and Eq. (142) must occur with $t < t_c$, we arrive at the conclusion of Eq. (133). Also, for ν to be real throughout the time period between when the mode $k > k_0$ leaves the horizon and the mode k_0 leaves the horizon, we must have

$$\nu(m_1) > \sqrt{2\varepsilon_{k_0}(N_k - N_{k_0})}\frac{m}{H(k_0)} \quad \text{for } k > k_0. \tag{143}$$

which follows from the justification of Eq. (125) with $\nu \to \nu(m_1)$. This sets an upper bound of k to be at

$$k < k_0 \exp\left[\frac{\nu^2}{2\varepsilon_{k_0}}\frac{H^2(t_{k_0})}{m^2}\right]. \tag{144}$$

The k bound coming from $\chi(t_k)$ being connected to $\chi(t_{k_0})$ through a de Sitter space solution is the same as Eq. (68).

3. APPLICATIONS

3.1. Improvement of the Axion Blue Isocurvature Scenario [1]

In this section we apply our theorems to the scenario of [1] and compute $O(1/(n-4)^2)$ corrections to their blue spectrum.

3.1.1. A Review of the Axion Blue Isocurvature Scenario [1]

We begin by reviewing [1]. They consider a renormalizable superpotential

$$W = h(\Phi_+\Phi_- - F_a^2)\Phi_0 \tag{145}$$

where the subscripts on Φ indicate $U(1)$ global charges. The F-term potential is

$$V_F = h^2|\Phi_+\Phi_- - F_a^2|^2 + h^2(|\Phi_+|^2 + |\Phi_-|^2)|\Phi_0|^2. \tag{146}$$

A flat directions of V_F exists along

$$\Phi_+\Phi_- = F_a^2 \qquad \Phi_0 = 0. \tag{147}$$

Their soft-SUSY breaking terms are assumed to be

$$V_{soft} = m_+^2|\Phi_+|^2 + m_-^2|\Phi_-|^2 + m_0^2|\Phi_0|^2 \tag{148}$$

where $m_i = O(\text{TeV})$. The Kaehler potential induced potential is

$$V_K = c_+H^2|\Phi_+|^2 + c_-H^2|\Phi_-|^2 + c_0H^2|\Phi_0|^2 \tag{149}$$

where $c_{+,-,0}$ are positive $O(1)$ constants. In addition to these, there can be H induced trilinear terms which can spoil the flat direction. Hence, they assume that the inflaton sector can be arranged to have $H \ll F_a$ such that the flat directions are only lifted by the quadratic terms.

Looking along the flat direction of Eq. (147) (more explicitly, setting $\Phi_0 = 0$), they have the effective potential being

$$V \approx h^2|\Phi_+\Phi_- - F_a^2|^2 + c_+H^2|\Phi_+|^2 + c_-H^2|\Phi_-|^2. \tag{150}$$

During inflation, the minimum of $\Phi_\pm$ lies at

$$|\Phi_\pm^{\min}| \approx \left(\frac{c_\mp}{c_\pm}\right)^{1/4} F_a. \tag{151}$$

They assume $\Phi_\pm$ starts out away from the minimum with a magnitude larger than this and approaches the minimum during inflation. This implies the $U(1)$ symmetry is broken during inflation. Hence, there will be a linear combination of the phases of $\Phi_\pm$ which will be the Nambu-Goldstone boson associated with the broken $U(1)$. Hence, they make a judicious sigma model parameterization

$$\Phi_\pm \equiv \frac{\varphi_\pm}{\sqrt{2}} \exp\left(i\frac{a_\pm}{\sqrt{2}\varphi_\pm}\right) \tag{152}$$

where $\varphi_\pm$ and $a_\pm$ are real. For our explanation later, keep in mind that $\varphi_\pm$ and $a_\pm$ are four distinct dynamical degrees of freedom.

The potential in the new variable is

$$V \approx -h^2 F_a^2 \varphi_+ \varphi_- \cos\left[\frac{a_+\varphi_- + a_-\varphi_+}{\sqrt{2}\varphi_+\varphi_-}\right] + h^2 F_a^4 + \frac{1}{4}h^2\varphi_-^2\varphi_+^2 \\ +\frac{1}{2}c_+ H^2\varphi_+^2 + \frac{1}{2}c_- H^2 \varphi_-^2 \tag{153}$$

For any fixed value of $\varphi_\pm$, one can see that a linear combination of $a_\pm$ will be the Nambu-Goldstone boson with an approximately flat direction. That axion combination called a is determined in their equation 15:

$$a = \frac{\varphi_+}{\sqrt{\varphi_+^2+\varphi_-^2}}a_+ - \frac{\varphi_-}{\sqrt{\varphi_+^2+\varphi_-^2}}a_- \tag{154}$$

which is not to be confused with the scale factor (whenever it is not clear from the context, we will add the subscript a_{scale} to denote the metric scale factor). The orthogonal combination

$$b = \frac{\varphi_-}{\sqrt{\varphi_+^2+\varphi_-^2}}a_+ + \frac{\varphi_+}{\sqrt{\varphi_+^2+\varphi_-^2}}a_- \tag{155}$$

has a potential

$$V_b = -h^2 F_a^2 \varphi_+\varphi_- \cos\left(\frac{\sqrt{\varphi_+^2+\varphi_-^2}}{\varphi_+\varphi_-}b\right). \tag{156}$$

They assume φ_+ is initially large (of order of M_P). Note that since $\varphi_+\varphi_- \approx 2F_a^2$ along the flat direction, if $\varphi_+(t_i) \sim M_P$, then

$$\varphi_-(t_i) \sim \frac{F_a^2}{M_P} \ll \varphi_+. \tag{157}$$

Hence, the field b during this time near t_i has a mass of order $h\varphi_+ \sim hM_P \gg H$ (for $F_a \lesssim M_P$) and thus is assumed to be settled to the minimum of $b = 0$ during inflation. The mass squared matrix of $\varphi_\pm$ also says thatφ_+ mass squared is of order of H^2 such that it can be dynamical. Since b is sitting at $b = 0$, Eq. (155) implies

$$\delta a_- \approx -\frac{\varphi_-}{\varphi_+}\delta a_+. \tag{158}$$

Hence, the angles appearing in Eq. (152) are

$$\delta\theta_+ \equiv \frac{\delta a_+}{\sqrt{2}\varphi_+} \tag{159}$$

and

$$\delta\theta_{-} \equiv \frac{\delta a_{-}}{\sqrt{2}\varphi_{-}} \tag{160}$$

Because of Eq. (157), we have the fluctuations in the a field being

$$a \approx a_{+} = \theta_{+}\sqrt{2}\varphi_{+}. \tag{161}$$

They therefore claim that the quantum fluctuations of

$$S = \frac{2\delta a}{a} \tag{162}$$

is frozen upon horizon departure. They assume a is massless and assign δa an amplitude of $H/(2\pi)$ where H is approximately constant during inflation. With that reasoning, they write

$$S_{\text{amplitude}} = \frac{H}{\sqrt{2}\pi\varphi_{+}\theta_{+}}. \tag{163}$$

It is important to note here that as far as identifying δa with $H/(2\pi)$ is concerned, they are neglecting the kinetic term induced mass of the a field which is what we are going to correct below.

Note φ_{+}decreases as a function of time as

$$\varphi_{+}(t) \approx \varphi_{+}(t_1)\left(\frac{a_{\text{scale}}(t)}{a_{\text{scale}}(t_1)}\right)^{-\frac{3}{2}+\nu} \tag{164}$$

with

$$\nu = \frac{3}{2}\sqrt{1 - \frac{4}{9}\frac{m_{++}^2}{H^2}} \tag{165}$$

where m_{++}^2 is the mass squared of the φ_{+} field and t_1 being a fiducial time. Since horizon exit condition gives

$$\frac{a_{\text{scale}}(t_1)}{a_{\text{scale}}(t)} = \left(\frac{k}{a_{\text{scale}}(t_1)H(t_1)}\right)^{-\varepsilon-1} \tag{166}$$

which gives

$$S_{\text{amplitude}} = \frac{H(t_1)}{\sqrt{2}\pi\varphi_{+}(t_1)\theta_{+}}\left(\frac{k}{a_{\text{scale}}(t_1)H(t_1)}\right)^{\left(\frac{3}{2}-\nu\right)(1+\varepsilon)-\varepsilon}. \tag{167}$$

This is indeed a blue spectrum in the limit $\nu \to 0$ (i.e. for $m/H \to 3/2$).

3.1.2. Improvement

According to our theorem 1, what needs to be frozen is $\delta a/a$. Since a is decaying with time (in contrast to the standard axion scenario), δa cannot be massless. On the other hand, [1] assigns

a massless spectral amplitude to δa of $H/(2\pi)$ (see Eq. (163) above). The goal of this subsection is to remedy this.

Although the action is a bit simpler if we choose other σ-model parameterization, we will consider the explicit axion parameterization $\{a, b, \varphi_\pm\}$ considered in [1] for transparency in connecting with this work. The kinetic term term is

$$
\begin{aligned}
K &= |\partial\Phi_+|^2 + |\partial\Phi_-|^2 \qquad (168)\\
&= \frac{1}{4}(\partial a)^2 + \frac{1}{4}(\partial b)^2 + \\
&\quad -\frac{a}{2}\partial_\mu a\left[\sin^2\gamma\partial^\mu \ln\varphi_- + \cos^2\gamma\partial^\mu \ln\varphi_+\right] + b\partial_\mu a\left[\partial^\mu \ln\varphi_- - \partial^\mu \ln\varphi_+\right]\sin\gamma\cos\gamma \\
&\quad -\frac{b}{2}\partial_\mu b\left[\cos^2\gamma\partial^\mu \ln\varphi_- + \sin^2\gamma\partial^\mu \ln\varphi_+\right] + \\
&\quad \partial_\mu\varphi_-\partial^\mu\varphi_-\left[\frac{1}{2} + \frac{a^2}{4\varphi_+^2}\cos^2\gamma\sin^2\gamma - \frac{ab}{\varphi_+^2}\cos^3\gamma\sin\gamma + \frac{b^2}{4\varphi_+^2}\left(\cos^2\gamma + 4\sin^2\gamma\right)\frac{\cos^4\gamma}{\sin^2\gamma}\right] + \\
&\quad \frac{1}{2}\partial_\mu\varphi_-\partial^\mu\varphi_+\frac{1}{\varphi_+^2}\left[-2ab\cos^4\gamma + a^2\cos^3\gamma\sin\gamma - 3b^2\cos^3\gamma\sin\gamma + 2ab\cos^2\gamma\sin^2\gamma\right] + \\
&\quad \partial_\mu\varphi_+\partial^\mu\varphi_+\left[\frac{1}{2} + \frac{a^2}{4\varphi_+^2}\cos^4\gamma + \frac{ab}{\varphi_+^2}\cos^3\gamma\sin\gamma + \frac{b^2}{4\varphi_+^2}\left(\sin^2\gamma + 4\cos^2\gamma\right)\sin^2\gamma\right] \qquad (169)
\end{aligned}
$$

where

$$\cos\gamma \equiv \frac{\varphi_+}{\sqrt{\varphi_+^2 + \varphi_-^2}}. \tag{170}$$

Because b and φ_- have masses much larger than the expansion rate H, they will sit at the minimum. Although the global minimum of this potential is at

$$b = 0, \;\; \varphi_+|_{min} = \sqrt{2}\sqrt{\frac{\sqrt{c_-}}{\sqrt{c_+}}F_a^2 - \frac{c_-}{h^2}H^2}, \quad \varphi_-|_{min} = \sqrt{2}\sqrt{\frac{\sqrt{c_+}}{\sqrt{c_-}}F_a^2 - \frac{c_+}{h^2}H^2} \tag{171}$$

assuming F_a is sufficiently larger than H, when φ_+ is displaced from its global minimum, the φ_- will sit at the φ_- variation minimum of

$$\varphi_- = \frac{2F_a^2}{\varphi_+}\frac{1}{1 + 2\frac{c_-}{h^2}\frac{H^2}{\varphi_+^2}} \tag{172}$$

which varies as a function of time because φ_+ varies as a function of time. On the other hand, because its mass is heavy, its fluctuations are not dynamically important.

For later discussion of the dynamics of the φ_+ and a, it is useful to note that the a field here corresponds to coset space parameterization of the spontaneously broken PQ symmetry: $U(1)_{PQ}$ defined as

$$\Phi_\pm \to e^{\pm i\theta}\Phi_\pm. \tag{173}$$

Because $U(1)_{PQ}$ breaking VEV is considered in the dynamics (i.e. $\varphi_\pm$ dynamics), the full dynamics is characterized by an unbroken $U(1)_{PQ}$. $U(1)_{PQ}$ gives rise to an exactly conserved current J^μ_{PQ} that is conveniently expressed in terms of $a_\pm$:

$$J^\mu_{PQ} = \varphi_- \partial^\mu a_- - \varphi_+ \partial^\mu a_+ + a_+ \partial^\mu \varphi_+ - a_- \partial^\mu \varphi_-. \tag{174}$$

This current is one piece of crucial information not discussed in [1].

Integrating out b (i.e. set $b = 0$ in Eq. (169)), we have

$$\begin{aligned} K \approx\ & \frac{1}{4}(\partial a)^2 - \frac{a}{2}\partial_\mu a \left[\sin^2\gamma \partial^\mu \ln \varphi_- + \cos^2\gamma \partial^\mu \ln \varphi_+\right] + \\ & \partial_\mu \varphi_- \partial^\mu \varphi_- \left[\frac{1}{2} + \frac{a^2}{4\varphi_+^2}\cos^2\gamma \sin^2\gamma\right] + \\ & \frac{1}{2}\partial_\mu \varphi_- \partial^\mu \varphi_+ \frac{1}{\varphi_+^2}\left[a^2 \cos^3\gamma \sin\gamma\right] + \\ & \partial_\mu \varphi_+ \partial^\mu \varphi_+ \left[\frac{1}{2} + \frac{a^2}{4\varphi_+^2}\cos^4\gamma\right]. \end{aligned} \tag{175}$$

Next, we will focus on the dynamics while $\varphi_+ \gg \varphi_-$ and $\varphi_+ \gg a$ where we will be able to integrate out φ_-. Since $\gamma \ll 1$ in this regime, we can then drop the $\sin\gamma$ terms to write

$$\begin{aligned} K \approx\ & \frac{1}{4}(\partial a)^2 - \frac{a}{2\varphi_+}\partial_\mu a \partial^\mu \varphi_+ + \\ & \frac{1}{2}\partial_\mu \varphi_- \partial^\mu \varphi_- + \left[\frac{1}{2} + \frac{a^2}{4\varphi_+^2}\right]\partial_\mu \varphi_+ \partial^\mu \varphi_+. \end{aligned} \tag{176}$$

Next, since the mass of φ_- is of the order

$$h^2 \varphi_+^2 \gg H^2, \tag{177}$$

we can integrate out φ_-, leaving a and φ_+ as only dynamical degrees of freedom. Note also that from the consideration of the kinetic term Eq. (176), the axion is no longer shift invariant as long as $\partial^\mu \varphi_+ \neq 0$. Note also that the simplification embodied in Eq. (176) breaks down when φ_+ is near its global minimum of Eq. (171). Also, in order to stabilize φ_+ at the minimum, we have implicitly assumed

$$F_a^2 > \frac{\sqrt{c_- c_+}}{h^2} H^2 \tag{178}$$

and in order to decouple φ_- and b at the minimum, we have assumed

$$F_a^2 \gg \frac{\sqrt{c_- c_+}}{h^2(c_- + c_+)} H^2. \tag{179}$$

The equation of motion for a and φ_+ are

$$\Box a - a\frac{\Box\varphi_+}{\varphi_+} = 0 \tag{180}$$

$$\Box\varphi_+ + \frac{1}{\sqrt{g}}\partial_\mu\left[\frac{a^2}{2\varphi_+^2}\sqrt{g}\partial^\mu\varphi_+\right] - \frac{1}{2\varphi_+}\frac{1}{\sqrt{g}}\partial_\mu\left[\sqrt{g}a\partial^\mu a\right] + \partial_{\varphi_+}V = 0 \tag{181}$$

Next, consider the background and linearized equations for a and φ_+ with the φ_- and b sitting at its minimum

$$b = 0 \tag{182}$$

$$\varphi_- = \frac{2F_a^2}{\varphi_+}\left[\frac{1}{1+2\frac{c_-}{h^2}\frac{H^2}{\varphi_+^2}}\right] \tag{183}$$

leading to the approximate potential of

$$V|_{\varphi_-,b=min} = \frac{H^2}{2}\left(c_+\varphi_+^2 + \frac{4c_-h^2F_a^4}{h^2\varphi_+^2+2c_-H^2}\right). \tag{184}$$

With this truncation, we have the conserved $U(1)_{PQ}$ charge density being

$$J^0_{PQ} \approx -(\frac{4F_a^4}{\varphi_+^4}+1)\,(\varphi_+\dot{a}_+ - a_+\dot{\varphi}_+)\,. \tag{185}$$

Because of the spacetime expansion, this charge density dilutes away: i.e.

$$\frac{a_+(t)}{a_+(t_1)} \approx \frac{\varphi_+(t)}{\varphi_+(t_1)} \tag{186}$$

to exponential accuracy. Note that during the early period of inflation when $\varphi_+ \gg F_a$, we have $a \approx a_+$. Eq. (186) and theorem 1 explains why during this time period

$$\frac{\delta a}{a(t)} \approx \frac{\varphi_+(t_1)}{a_+(t_1)}\frac{\delta a}{\varphi_+(t)} = \text{constant.} \tag{187}$$

In other words, $\delta a/\varphi_+$ is frozen because it behaves like $\delta a/a$ because of $U(1)_{PQ}$ even though the numerator and the denominator are independent degrees of freedom for which theorem 1 would not always apply. For the complete spectrum calculation, it is better to rewrite Eq. (186) in terms of $a(t)$:

$$\frac{a(t)}{a(t_1)} = \frac{\sqrt{\frac{4F_a^4}{\varphi_+^2(t)}+\varphi_+^2(t)}}{\sqrt{\frac{4F_a^4}{\varphi_+^2(t_1)}+\varphi_+^2(t_1)}}. \tag{188}$$

Let's consider the magnitude of the initial charge that becomes diluted to assess the accuracy of Eq. (186):

$$Q \equiv \int d^3x a^3(t_0) J^0_{PQ}(t_0). \tag{189}$$

If we assume all dynamical scales are tied to H, we can estimate

$$Q \approx O(H)\varphi_+(t_0)a_+(t_0)a^3_{\text{scale}}(t_0) \tag{190}$$

and the diluted charge density at any time t after the initial time t_0 is

$$J^0_{PQ}(t) \sim O(H)\varphi_+(t_0)a_+(t_0)\frac{a^3(t_0)}{a^3(t)}. \tag{191}$$

This translates into an Eq. (186) accuracy of

$$\text{accuracy} \equiv O(e^{-3\Delta N}) \tag{192}$$

where $\Delta N > 0$ is the efold time from the beginning of the inflation t_0 to the time of horizon exit of a given mode. For example, if we want an accuracy of $O(\varepsilon)$, we only need the beginning of inflation and the time of horizon exit for the longest wavelength mode of phenomenological interest labeled by k_{min} to be separated by the efold number of

$$\Delta N \sim -\frac{1}{3}\ln\varepsilon \tag{193}$$

before we can set t_1 of Eq. (186) to $t_{k_{\text{min}}}$ while assuming for $t > t_{k_{\text{min}}}$. For $\varepsilon = 0.01$, one only gives up about 1 efolding. During this one efolding, the $\varphi_+(t)$ decays compared to its value at the beginning of inflation $\varphi_+(t_0)$. Although its exact trajectory is initial condition dependent, one can estimate a lower bound on its decay as long as the initial conditions have $\dot{\varphi}_+ \lesssim -c_+H\varphi_+$ which is an attractor solution:

$$\varphi_+(t_{k_{\text{min}}}) = \varphi_+(t_0)\left(\frac{a_{\text{scale}}(t_0)}{a_{\text{scale}}(t_{k_{\text{min}}})}\right)^{\frac{3}{2}-\nu} \tag{194}$$

Since $\varphi_+(t_0) \lesssim M_p$, this sets an upper bound on $\varphi_+(t_{k_{\text{min}}})$ for the validity of the analytic computation:

$$\frac{\varphi_+(t_{k_{\text{min}}})}{M_P} \lesssim (\text{accuracy})^{\left(\frac{1}{2}-\frac{\nu}{3}\right)}. \tag{195}$$

To compute the δa correlator using theorem 3, we need the mass of a. We can obtain the mass by writing the equation of motion for a, neglecting the small corrections proportional to c_-:

$$\left(\Box - \frac{\Box\varphi_+}{\varphi_+}\right)a = 0 \tag{196}$$

$$\Box\varphi_+ + \frac{1}{\sqrt{g}}\partial_\mu\left[\frac{a^2}{2\varphi_+^2}\sqrt{g}\partial^\mu\varphi_+\right] - \frac{1}{2\varphi_+}\frac{1}{\sqrt{g}}\partial_\mu\left[\sqrt{g}a\partial^\mu a\right] + c_+H^2\varphi_+ = 0. \tag{197}$$

The fact that a acquires a time dependent mass is important because that means that the decay of the a field due to the mass will stop after a finite time period: i.e. because a is a NG boson, its mass will shut off in the vacuum. Hence, we see that in the limit of $a/\varphi_+ \ll 1$, we find

$$\left(\Box + c_+ H^2\right) a \approx 0 \tag{198}$$

$$\left(\Box + c_+ H^2\right) \varphi_+ \approx 0. \tag{199}$$

There is a remarkable symmetry in this limit because of $U(1)_{PQ}$ as explained in Eq. (186). The mismatch between δa and $\delta\varphi_+$ coming from Eq. (196) is suppressed by $a/\varphi_+ \ll 1$.

Now, once φ_+ rolls to the minimum of Eq. (184), terms proportional to c_- dropped in Eq. (197) will become active allowing

$$\varphi_+ = \text{constant} = \sqrt{2}\sqrt{\frac{\sqrt{c_-}}{\sqrt{c_+}} F_a^2 - \frac{c_-}{h^2} H^2}. \tag{200}$$

We will call the time when φ_+ settles down to this value such that

$$\left.\frac{\Box\varphi_+}{\varphi_+}\right|_{t=t_c} \ll H^2 \tag{201}$$

At time t_c, a will become massless because it is a NG boson. On the other hand φ_+mass does not shut off even after φ_+ reaches its minimum. Hence, it can be shown (similarly as in [98]) that $\delta\varphi_+$ keeps decreasing while δa freezes out. Indeed, this decay of the φ_+ perturbations which make it negligible is similar to the reason why one uses a Yukawa interaction to generate observable isocurvature perturbations in the fermionic isocurvature perturbations of [101]: i.e. the non-interaction piece has a blue spectrum of $n \approx 7$.

Since the axion mass seen in Eq. (180) is time dependent, corollary 2 is useful to compute the isocurvature spectrum. We find

$$\Delta_s^2(k) = \omega_a^2 \left(\frac{2^{2\nu+1}|\Gamma(\nu(\sqrt{c_+}H))|^2}{\pi}\right) \left(\frac{H(t_{k_0})}{2\pi a(t_{k_0})}\right)^2 \left(\frac{k}{k_0}\right)^{3-2\nu(\sqrt{c_+}H)-2\varepsilon_{k_0}+O(\varepsilon_{k_0})g_0(m/H)} \tag{202}$$

and the QCD axion fractional density from coherent oscillations can be estimated for $F_a \ll 10^{17}$ GeV as (see e.g. Eq. (14) of [77])

$$\omega_a \equiv \frac{\Omega_a}{\Omega_{cdm}} \approx W_a \left(\frac{a(t_c)}{\sqrt{2}\sqrt{\varphi_+^2(t_c) + \varphi_-^2(t_c)}}\right)^2 \left(\frac{\sqrt{2}\sqrt{\varphi_+^2(t_c) + \varphi_-^2(t_c)}}{10^{12}\ \text{GeV}}\right)^{n_{PT}} \tag{203}$$

where

$$n_{PT} \equiv 1.19 \qquad W_a \equiv \frac{3}{2} \tag{204}$$

and we have used $\Omega_{cdm}h^2 = 0.12$. Here k_0 is fixed fiducial wave vector and t_{k_0} is the time when that mode leaves the horizon. Furthermore, we can use Eq. (171) to set

$$\sqrt{\varphi_+^2(t_c) + \varphi_-^2(t_c)} \approx F_a \sqrt{\frac{2}{\sqrt{c_- c_+}}(c_- + c_+)} \tag{205}$$

corresponding to the minimum assuming $F_a \gg H$. This gives a CDM fraction of

$$\omega_a \approx W_a \theta_+^2(t_{k_0}) \left(\frac{2F_a \sqrt{\frac{1}{\sqrt{c_- c_+}}(c_- + c_+)}}{10^{12}\ \mathrm{GeV}} \right)^{n_{PT}} \tag{206}$$

which saturates the relic bound

$$F_a \sim \theta_+^{-2/n_{PT}}(t_{k_0}) \times 10^{12}\ \mathrm{GeV}. \tag{207}$$

Unlike in the ordinary axion scenario where $H/(2\pi F_a)$ sets the variance of the effective classical initial condition for $\theta_+(t_{k_0})$, here $H/(2\pi\varphi_+(t_{k_0})) \ll H/(2\pi F_a)$ sets the variance.[9] Because the axion is a non-thermal dark matter field after the end of inflation and because Eq. (201) is expected to be continually satisfied after the end of inflation due to the weak time dependence of Eq. (200), we expect Eq. (202) to be a good approximation to the final blue isocurvature spectrum in the model of [1].

For the validity of Eq. (202) coming from corollary 2, the wave vector k must lie in the range

$$k_{\min} < k < k_{\max} \tag{208}$$

where $k_{\min}$ and $k_{\max}$ have parameter dependent constraints which we now discuss. In practice, $k_{\min}$ and $k_{\max}$ should be chosen to saturate the most stringent of the bounds listed in the corollary. Since Bunch-Davies initial conditions must be set up for the quantum fields after inflation starts, we must have $k_{\min} > a(t_b)H(t_b)$ (where t_b is the beginning of inflation). Since the mode exit time $t_k < t_c$ (where t_c is the time that the time dependent mass shifts):

$$k_{\min} < k_{\max} \lesssim k_{\min} \times (\mathrm{accuracy})^{\frac{1}{3}} \left(\frac{M_p}{F_a} \frac{1}{\sqrt{2}\left(\frac{c_-}{c_+}\right)^{1/4}} \right)^{\frac{2}{3-2\nu\left(\sqrt{c_+}H\right)}} \tag{209}$$

[9] Hence, even with $H \sim 10^{14}$ GeV, one can have $\varphi_+(t_{k_0}) \sim M_p$ and thus tune θ_+ as small as 10^{-5} without considerations of the variance. Of course, one can even go much smaller than the variance with 10% tuning as well.

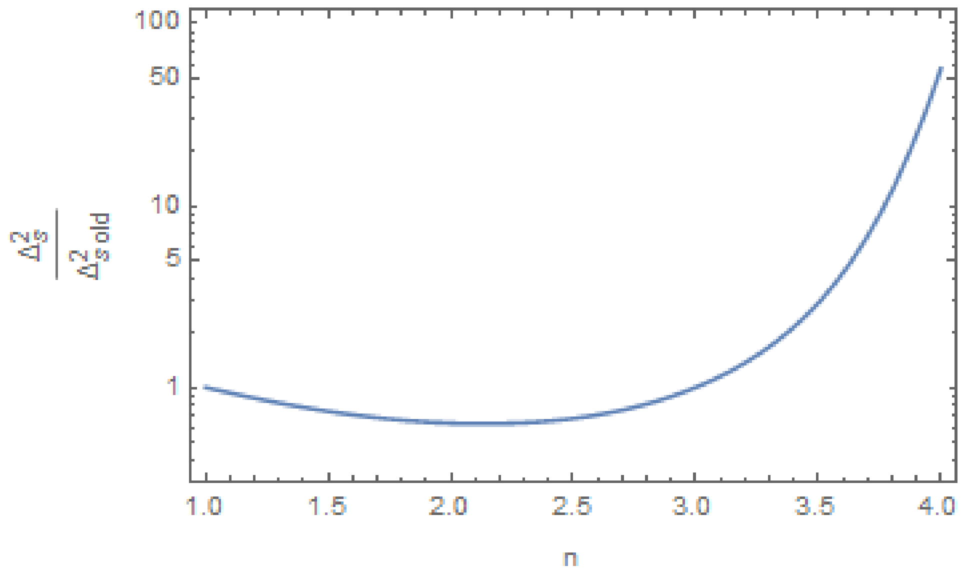

Figure 1: Comparison of the result of this paper with the result of [1] for the amplitude of $k = k_0$ (e.g. can be taken to be the CMB scale), neglecting small slow-roll parameter corrections of $O(\varepsilon)$. The enhancement is limited by the condition of Eq. (65). For $m/H = 1.497$,one can have $n \approx 3.8$ and an enhancement of about 11.

where we required $\varphi_+(t)$ to be smaller than M_p ("accuracy" is defined by Eq. (192) associated with the attractor assumption). If other constraints on k_{max} that we discuss below allows it, k_{max} can realistically become very close to saturating the upper bound. One can conveniently set "accuracy" equal to the analytic approximation accuracy f that one seeks, although in principle, they can be set independently. One of the important constraints related to f coming from Eq. (133) with $k = k_{\mathrm{max}}$ is

$$k_{\mathrm{max}} \ll k_0 f^{1/\nu} \left(\frac{\varphi_+(t_{k_0})}{\sqrt{2} F_a \left(\frac{c_-}{c_+}\right)^{1/4}} \right)^{\frac{2}{3-2\nu\left(\sqrt{c_+}H\right)}} . \tag{210}$$

Also, because one is using a scaling approximation about the fiducial wave vector k_0, there is a set of inflationary model dependent validity constraints. Given the complicated parametric dependences of these boundaries, we will summarize below and give an explicit example.

The main correction to the original isocurvature result of [1] (i.e. Eq. (21) after accounting for footnote 5 in [1]) in the blue spectral index region (i.e. before the break) is the spectral index dependent factor

$$R_s(c_+) \equiv \frac{\Delta_s^2(k)}{\Delta_{s\,\mathrm{old}}^2(k)} = \frac{2^{2\nu\left(\sqrt{c_+}H\right)-1} |\Gamma\left(\nu\left(\sqrt{c_+}H\right)\right)|^2}{\pi} \tag{211}$$

where one must keep in mind that ν cannot be exactly zero because of Eq. (65).[10] This is an interesting factor because it can give a factor of $O\left(1/(n-4)^2\right)$ enhancement for the blue spectrum due to the prefactor $|\Gamma(\nu)|^2$ while still satisfying Eq. (65). For example, with $m/H = 1.497$, we have $n \approx 3.8$, we obtain an enhancement factor of 11. To satisfy Eq. (65), we merely need to satisfy

$$0.09\frac{f}{\varepsilon_k} \gg 1$$

for inflationary scenarios with $\varepsilon_k = 10^{-4}$ even if one desires a percent accuracy of $f = 0.01$. Such enhancements are surprising because as $\nu \to 0$, one expects the quantum fluctuation amplitude of a to be smaller because its spacetime-curvature induced tachyonic growth is smaller. Indeed, one can easily check that $\langle \delta a_{nad} \delta a_{nad} \rangle$ indeed is smaller as its effective mass controlled by c_+ is increased. For the same parameters, the background field a also decreases as c_+ is increased. The asymptotic amplitude of $\langle \delta a_{nad} \delta a_{nad} \rangle$ associated with the Hankel function solution decreases less as $c_+ \to 3/2$ from below. This is a nontrivial difference between the quantization induced Hankel function versus the homogeneous mode function with slow-roll initial conditions. The comparison of the new result with the old one is given in Fig. 1.

Next, we will summarize the explicit form of the improved spectrum together with their validity conditions. Using Eq. (205) and Eq. (186), we can rewrite Eq. (202) as

$$\Delta_s^2(k) = 4P_{QCD}(c_+, c_-, F_a)\theta_+^2(t_{k_0}) \times \begin{cases} R_s(c_+)\left(\frac{H(t_{k_0})}{2\pi\varphi_+(t_{k_0})}\right)^2\left(\frac{k}{k_0}\right)^{3-2\nu(\sqrt{c_+}H)-2\varepsilon_{k_0}+O(\varepsilon_{k_0})g_0(\sqrt{c_+})} & k_{\min} < k < k_{\max} < k_c \\ \left(\frac{H(t_{k_0})}{2\pi\sqrt{2}F_a}\right)^2 \frac{\sqrt{c_+c_-}}{c_++c_-}\left(\frac{H(t_0)}{H(t_c)}\frac{k_0}{k_c}\right)^{-2\varepsilon_{k_c}}\left(\frac{k}{k_0}\right)^{-2\varepsilon_{k_c}} & k_c < k < k_e \end{cases} \tag{212}$$

$$R_s(c_+) \equiv \frac{2^{2\nu\left(\sqrt{c_+}H\right)-1}|\Gamma\left(\nu\left(\sqrt{c_+}H\right)\right)|^2}{\pi}$$

$$k_c \equiv k_0\left(\frac{\varphi_+(t_{k_0})}{\sqrt{2}F_a\left(\frac{c_-}{c_+}\right)^{1/4}}\right)^{\frac{2}{3-2\nu(\sqrt{c_+}H)}} \qquad \nu(x) = \frac{3}{2}\sqrt{1-\frac{4}{9}\frac{x^2}{H^2}} \tag{213}$$

[10] Note that if one wants to compute this in a numerical setting, one must be careful to tune the background field boundary condition $a(t_{\text{initial}})$ to keep the axion background field value at fiducial mode horizon crossing $a(t_{k_0})$ fixed for any fixed choice of k_0 and m. Also, note that as long as Eq. (65) is obeyed, the expansion of the Hankel function from which the $\Gamma(\nu)$ arises is a good approximation.

$$\frac{k_{\max}}{k_0} = \min\left\{ f^{1/\nu}\left(\frac{\varphi_+(t_{k_0})}{\sqrt{2}F_a\left(\frac{c_-}{c_+}\right)^{1/4}}\right)^{\frac{2}{3-2\nu(\sqrt{c_+}H)}}, \frac{k_{\min}}{k_0} f^{1/3}\left(\frac{M_p}{F_a}\frac{1}{\sqrt{2}\left(\frac{c_-}{c_+}\right)^{1/4}}\right)^{\frac{2}{3-2\nu(\sqrt{c_+}H)}} \right.$$
$$\left., \exp\left[\frac{\nu^2(\sqrt{c_+}H)}{2\varepsilon_{k_0}c_+}\right], \exp\left(\frac{f}{\varepsilon_{k_0}}\right)\right\} \tag{214}$$

$$\varepsilon_{k_0} = \frac{1}{2\Delta_\zeta^2(k_0)}\frac{1}{M_p^2}\left(\frac{H_{k_0}}{2\pi}\right)^2 \tag{215}$$

$$k_{\min} = \max\left\{k_0\exp\left(-\frac{f}{\varepsilon_{k_0}}\right),\ a(t_b)H(t_b)\right\} \tag{216}$$

$$P_{QCD}(c_+,c_-,F_a) \equiv W_a^2 2^{2n_{PT}-1}\left(\frac{c_-+c_+}{\sqrt{c_-c_+}}\right)^{n_{PT}}\left(\frac{F_a}{10^{12}\text{ GeV}}\right)^{2n_{PT}} \tag{217}$$

$$\frac{2}{3}\sqrt{2}\pi\left(\frac{k_{H_0}}{k_0}\right)^{-\frac{3}{2}+\nu+O(\varepsilon)g_0(m/H)} c_+\sqrt{\Delta_\zeta^2(k_{H_0})}\frac{M_p}{H}\frac{\theta_+(t_{k_0})}{f} < \frac{M_p}{\varphi_+(t_{k_0})}. \tag{218}$$

where $\Delta_\zeta^2(k_0) \approx 2.4\times10^{-9}$. We have also assumed $F_a \ll 10^{17}$ GeV for the background axion dark matter fraction. Some of the important background equation of motion simplifications allowing the analytic treatment come from

$$\frac{(c_+c_-)^{1/4}}{h}H \ll F_a \ll \varphi_+(t_{k_0}) \le \varphi_+(t_{k_{H_0}}) \approx \varphi_+(t_{k_0})\left(\frac{k_0}{k_{H_0}}\right)^{\frac{3}{2}-\nu(\sqrt{c_+}H)} \lesssim f^{\frac{1}{2}-\frac{\nu}{3}}M_p, \tag{219}$$

$$\frac{H}{2\pi\sqrt{2}\varphi_+(t_{k_0})} \ll \theta_+(t_{k_0}) \equiv \frac{a(t_{k_0})}{\sqrt{2}\varphi_+(t_{k_0})} \ll 1 \tag{220}$$

$$\frac{a(t_c)}{\varphi_+(t_c)} \ll 1 \longrightarrow \frac{c_+}{c_-} \ll \frac{1}{2\theta_+^2(t_{k_0})} - 1 \tag{221}$$

$$a(t_c) < a(t_{k_0}) \longrightarrow \sqrt{2}\frac{\sqrt{c_-+c_+}}{(c_-c_+)^{1/4}}F_a < \varphi_+(t_{k_0}) \tag{222}$$

where $k_{H_0} \approx a_0H_0\pi/2$ is the wave vector corresponding to the observable universe today. The vector $k_e = a(t_e)H(t_e)$ is the last wave vector to leave the horizon at the end of inflation and is typically at an unobservably small scale and is very model dependent. This gives the improved isocurvature spectrum (together with Eqs. (204) and (134)). The independent variables can be classified as axion-dependent parameters $\{F_a, \varphi_+(t_{k_0}), \theta_+(t_{k_0}), c_\pm\}$, inflation-dependent parameter $H(t_{k_0})$, and approximation scheme dependent parameters $\{f, k_0\}$. Since physics is obviously independent of different approximation scheme, the physical parameter space of this model is six

dimensional. The blue spectral index is controlled by only one parameter c_+ while the break in the spectrum is determined by $k_{\rm max}$

Eq. (220) gives the approximate condition for classical initial condition tuning of θ_+ assuming there being a quantum noise of $O(H/(2\pi))$ for the axion field because there is no enhanced symmetry that would protect the axion field from tadpole corrections. Although we do not address the tadpole issue here by an explicit computation, it is reasonable to expect that the tadpole quantum fluctuations can also significantly correct the background equation of motion (again in the absence of enhanced symmetries) later as φ_+ settles to its minimum. In that case, we should apply a condition

$$\frac{H(c_-c_+)^{1/4}}{4\pi F_a\sqrt{c_-+c_+}} \ll \theta_+(t_{k_0}) \tag{223}$$

which is stronger than Eq. (220) because of Eq. (222).

The isocurvature to adiabatic perturbation ratio is controlled by

$$\begin{aligned}\frac{\Delta_s^2(k)}{\Delta_\zeta^2(k)} &= P_{QCD}(c_+,c_-,F_a)\theta_+^2(t_{k_0})8\varepsilon_{k_0}\times \\ &\begin{cases} R_s(c_+)\left(\frac{M_p}{\varphi_+(t_{k_0})}\right)^2\left(\frac{k}{k_0}\right)^{3-2\nu(\sqrt{c_+}H)+4\varepsilon_{k_0}-2\eta_V} & k_{\rm min}<k,k_0<k_{\rm max}<k_c \\ \left(\frac{M_p}{\sqrt{2}F_a}\right)^2\frac{\sqrt{c_+c_-}}{c_++c_-}\left(\frac{H(t_0)}{H(t_c)}\frac{k_0}{k_c}\right)^{-2\varepsilon_{k_c}}\left(\frac{k}{k_0}\right)^{-2\eta_V+4\varepsilon_{k_0}} & k_c<k<k_e\end{cases}\end{aligned} \tag{224}$$

where the inflationary scalar spectral index is given by

$$n_s-1=2\eta_V-6\varepsilon \tag{225}$$

which can be used to phenomenologically specify η_V once n_s-1 and ε are fixed. The CDM fraction is given by

$$\omega_a \approx W_a\theta_+^2(t_{k_0})\left(\frac{2F_a\sqrt{\frac{1}{\sqrt{c_-c_+}}(c_-+c_+)}}{10^{12}\ \text{GeV}}\right)^{n_{PT}} < 1. \tag{226}$$

Because of Eq. (139), phenomenologically allowed parameters are in the regime of $\omega_a \ll 1$. If one accomplishes this with bringing down F_a, then Eq. (219) brings down H. This in turn brings down $\varphi_+(t_{k_0})$ because of Eq. (218).

Let us illustrate this formula with concrete parametric choices. We plot $\Delta_s^2(k)/\Delta_\zeta^2(k_0)$ in Fig. 2 for $c_+\in\{0.2,\ 1,\ 1.5\}$ with the rest of the parameters fixed at

$$c_-=0.9,\ \theta_+=0.04,\ F_a=7.9\times10^{10}\ \text{GeV},\ \varphi_+(t_{k_0})=8.3\times10^{-7}M_p,\ H=6\times10^{9}\ \text{GeV},\ f=0.2. \tag{227}$$

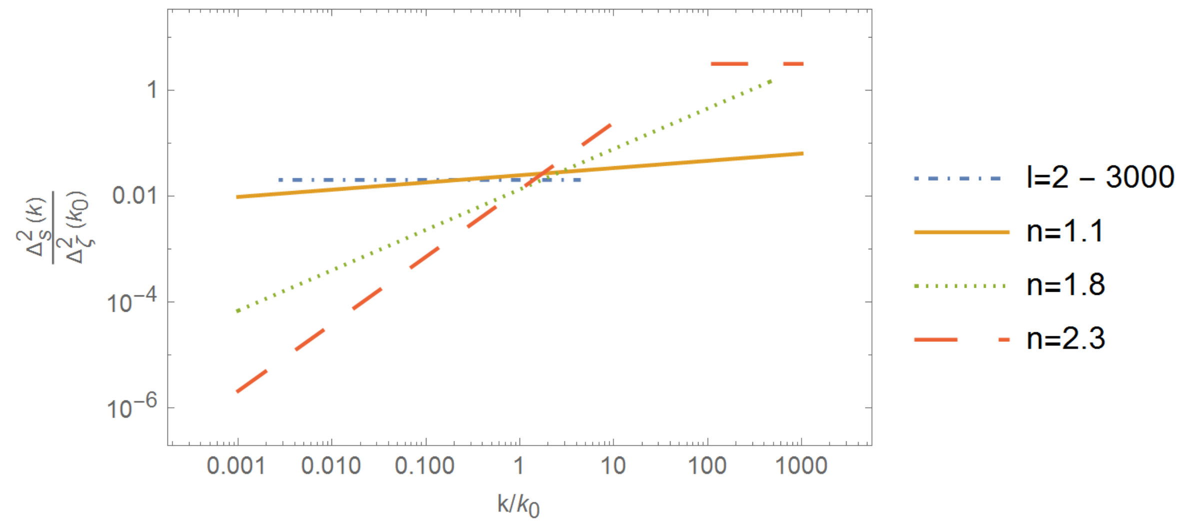

Figure 2: Illustration of the blue isocurvature spectra for several values of the parameter $c_+ \in \{0.2, 1, 1.5\}$ which map to isocurvature {spectral index, axion dark matter fraction ω_a} of $\{\{1.1, 4.73\times 10^{-4}\}, \{1.8, 4.06\times 10^{-4}\}, \{2.3, 4.13\times 10^{-4}\}\}$, respectively. The rest of the parameters are fixed as given in Eq. (227), where in particular, the dark matter fraction is. The gap in the dashed curve in the range $k/k_0 \in [10, 100]$ occurs as a result of breakdown of the analytic approximation associated with the fact that the effective time-dependent mass transition occurs before the modes classicalize. Note that the c_+ controls both the inflation induced Hubble scale mass for the non-vacuum axion as well as the amplitude of the spectra. That is why the three curves do not meet at a point. The dot-dashed CMB curve represents a flat spectrum with an amplitude of about 2% of the adiabatic spectrum extending from $l = 2-3000$ scale with $k_0 = 0.05$ Mpc^{-1}. Because of the transfer function suppressing isocurvature power relative to the adiabatic power on short length scales (an effect not shown here), the observational constraints on these example spectra are weak.

This set of c_+ map to {spectral index, axion dark matter fraction ω_a} of $\{\{1.1, 4.73\times 10^{-4}\}, \{1.8, 4.06\times 10^{-4}\}, \{2.3, 4.13\times 10^{-4}\}\}$, respectively. To compare with the approximate CMB length scales in the plot, we have fixed $k_0 = 0.05$ Mpc^{-1}.[11] The gap in the dashed curve in the range $k/k_0 \in [10, 100]$ occurs as a result of breakdown of the analytic approximation associated with the fact that the effective time-dependent mass transition occurs before the modes classicalize. The actual spectrum in this gap is not addressed by the techniques of this paper. A

[11] Given that this paper is a paper focused on analytic computation of the spectra, we leave more detailed numerical data fitting work to the future.

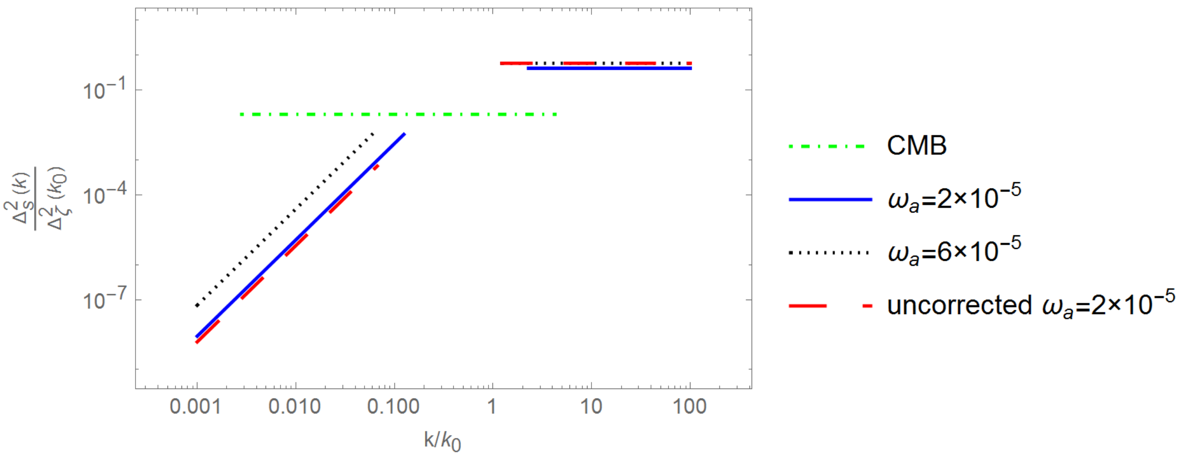

Figure 3: Isocurvature spectra for $F_a = \{6.8 \times 10^{10}\text{ GeV},\ 1.6 \times 10^{11}\text{ GeV}\}$ corresponding to the axion dark matter fraction of $\omega_a = \{2 \times 10^{-5}, 6 \times 10^{-5}\}$. The other parameters are fixed at values given in Eq. (228), and the resulting isocurvature spectral index is $n = 3.8$. The "uncorrected" label refers to the plot that would have been given based on the previous literature without Eq. (211). Note that without the correction, one would misidentify an experimental signal of $\omega_a = 2 \times 10^{-5}$ for that of $\omega_a = 6 \times 10^{-5}$. The CMB label and the gaps in the spectra are explained in Fig. 2. Note that parameters were chosen such as to stay close to being observable. It is easy to choose parameters such that this spectrum is unobservable in the foreseeable future.

similar breakdown of the assumptions leads to the termination of the dotted curve.

Next, we illustrate what happens when the isocurvature spectral index is very steep such that the correction factor of Eq. (211) becomes around a factor of 10. In Fig. 3, we plot the isocurvature spectra for $F_a = \{6.8 \times 10^{10}\text{ GeV},\ 1.6 \times 10^{11}\text{ GeV}\}$ corresponding to the axion dark matter fraction of $\omega_a = \{2 \times 10^{-5}, 6 \times 10^{-5}\}$, with the other parameters fixed at

$$c_+ = 2.235,\, c_- = 0.9,\, \theta_+ = 10^{-2},\, \varphi_+(t_{k_0}) = 10^{-7} M_p,\, H = 9 \times 10^9\text{ GeV},\, f = 0.7. \qquad (228)$$

The relatively extreme c_+ parametric choice gives an isocurvature spectral index of $n = 3.8$ and the spectral amplitude has an analytic approximation error at the level of around 70%.[12] It is remarkable that dark matter fraction ω_a as small as those considered in Figs. 2 and 3 can generate potentially observable effects in cosmology.

[12] A smaller choice of f (corresponding to a smaller approximation error) leads to the spectrum not being computable analytically in the interesting region.

Another question one might have is how large H can be in these scenarios, since H controls the amplitude of the tensor perturbations that may be observable by future experiments [102–107]. Unlike in the the ordinary axion situation, the effective PQ symmetry breaking VEV is much larger throughout the course of the isocurvature field evolution. This means that $H/(\text{time dependent PQ symmetry breaking VEV})$ which controls the effective amplitude of the isocurvature is much smaller for observable k region for the same value of F_a which controls the axion dark matter fraction ω_a. The parametric tension still arises as we will now see. Combining the stabilization constraints of φ_+ and b (i.e. Eqs. (178) and (179)) with the dark matter bound of Eq. (226) gives

$$W_a\theta_+^2(t_{k_0})\left(\frac{2H/h}{10^{12}\text{ GeV}}\right)^{n_{PT}} \ll \omega_a < 1. \tag{229}$$

Another constraint comes from the isocurvature bound of Eq. (224), where we restrict to the relevant subset of constraint

$$\frac{\Delta_s^2(k_0)}{\Delta_\zeta^2(k_0)} < \alpha_{k_0} \tag{230}$$

Both the dark matter bound of Eq. (229) and the isocurvature bound of Eq. (230) can be satisfied for sufficiently small $\theta_+(k_0)$. To find the minimum $\theta_+(k_0)$ allowed by the other constraints, note Eq. (221) (coming from decoupling axion mixing) and Eq. (223) (coming from the neglect of quantum tadpoles), and Eq. (220) imply

$$\frac{H^2}{8\pi^2}\frac{\sqrt{c_-c_+}}{F_a^2(c_-+c_+)} \ll 2\theta_+^2(t_{k_0}) \ll \min\left\{\frac{c_-}{c_+},2\right\}. \tag{231}$$

Hence, there is a minimum c_- for which this can be satisfied:

$$2 > \frac{(c_-)_{\text{min}}}{c_+} > \frac{1}{24}\left(2^{2/3}\left(Y(H/F_a)\right)^{1/3} + \frac{32}{\left(\frac{1}{2}Y(H/F_a)\right)^{1/3}} - 16\right) \tag{232}$$

$$Y(H/F_a) \equiv \frac{27H^4}{\pi^4F_a^4} + \frac{3H^2\sqrt{768\pi^4+81H^4/F_a^4}}{\pi^4F_a^2} + 128 \tag{233}$$

Putting this into Eq. (231) gives

$$\min(\theta_+^2) = \mu\left(\frac{F_a}{H}\right) \equiv \frac{1}{48}\left(2^{2/3}\left(Y(H/F_a)\right)^{1/3} + \frac{32}{\left(\frac{1}{2}Y(H/F_a)\right)^{1/3}} - 16\right). \tag{234}$$

where the right hand side is valid whenever it is smaller than unity. Now, the isocurvature bound in the form of Eq. (230) combined with Eq. (234) give

$$W_a^2 2^{2n_{PT}-1}\left(\frac{2\mu\left(\frac{F_a}{H}\right)+1}{\sqrt{\mu\left(\frac{F_a}{H}\right)}}\right)^{n_{PT}}\left(\frac{F_a}{10^{12}\text{ GeV}}\right)^{2n_{PT}}\mu\left(\frac{F_a}{H}\right)8\varepsilon_{k_0}R_s(c_+)\left(\frac{M_p}{\varphi_+(t_{k_0})}\right)^2 < \alpha_{k_0}. \tag{235}$$

The F_a dependent function on the left hand side of Eq. (235) is a monotonically increasing function of F_a and it is also a monotonically increasing function of H. Hence, to maximize H, we need to determine the smallest F_a that is allowed by our constraints.

One blunt constraint comes from the fact that μ is supposed to be a small angle. Since the small angle squared $\mu(F_a/H)$ is a monotonically decreasing function of F_a, one can find the smallest F_a allowed by the small angle assumption by solving $\mu(F_1/H) = 1$:

$$\frac{F_1}{H} \approx 0.05. \tag{236}$$

Another constraint comes from combining Eq. (229) with Eq. (234). Defining $F_2 = \min(F_a)$ subject to this constraint, one finds numerically

$$\frac{F_2}{H} \approx 0.2 \tag{237}$$

which is larger than F_1/H. One also finds numerically that $h \gtrsim 0.3$ up to the perturbative limit in this corner of allowed parametric region. Because the F_a dependence on H bound is weak, there is only a small shift in using F_2 versus F_1. In any case, since F_2 is a stronger constraint, we set $F_a = F_2$ in Eq. (235) and need to solve for the H upper bound. Note that setting $F_a = F_2$ corresponds to a $\min\left(\theta_+^2\right) = 0.04$ and $c_- = 0.08c_+$.[13]

To solve for the H upper bound, we still need to set c_+. Note although $c_+ \approx 1.3795$ gives the smallest Eq. (211) that shows up in Eq. (235), we must still check the constraints coming from Eq. (219). Eq. (178) part of the constraint in this parametric corner can be written explicitly as

$$\frac{(c_+c_-)^{1/4}}{h} H \ll F_a \longrightarrow \frac{(2\mu(0.2))^{1/4}\sqrt{c_+}}{h} \ll 0.2 \tag{238}$$

where one sees the explicit H independence because of Eq. (237). Another can be written as

$$\frac{\varphi_+(t_{k_0})}{M_p} \lesssim f^{\frac{1}{2}-\frac{\nu}{3}} \left(\frac{k_0}{k_{H_0}}\right)^{-\frac{3}{2}+\nu(\sqrt{c_+}H)}. \tag{239}$$

Finally, perturbativity requires

$$h < \sqrt{4\pi}. \tag{240}$$

[13] Note that this does not mean that the smallest θ_+^2 for all parts of the parameter space is 0.04. It is only when F_a is minimized subject to the constraints discussed, do we have this minimum on θ_+^2 .

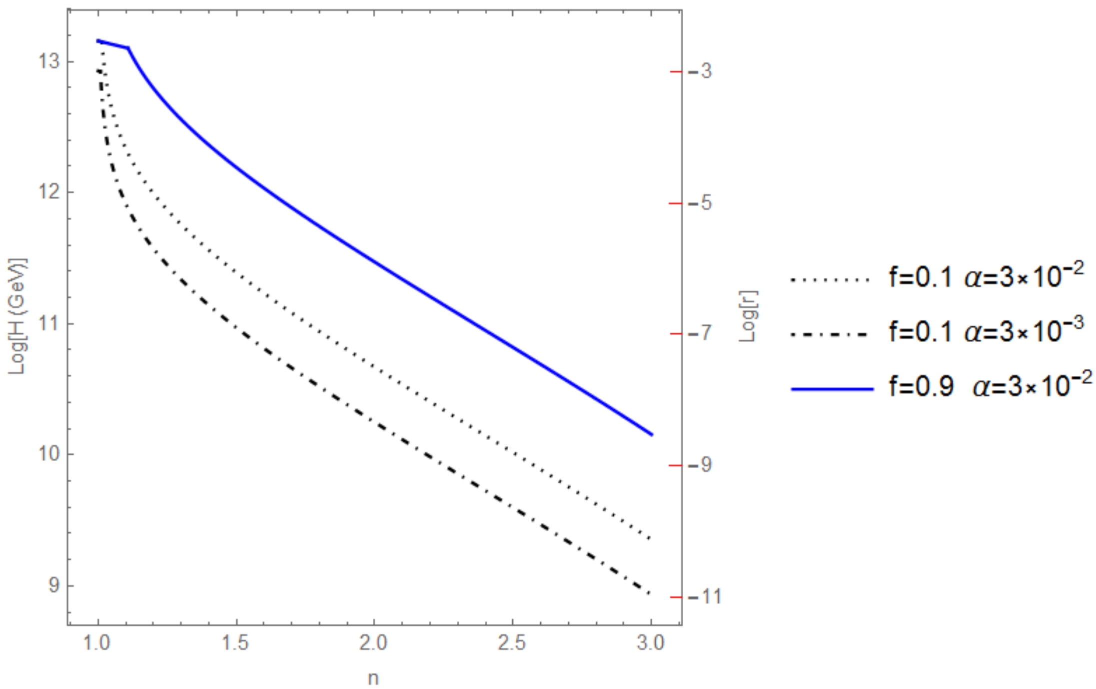

Figure 4: Maximum H as a function of the isocurvature spectral index n. On the right axis, the tensor-to-scalar power spectra ratio r is plotted. The break feature for small n in the $f = 0.9$ case corresponds to the situation where Eq. (239) becomes more important than the constraints coming from Eqs. (242) and (243)

These three constraints constrain $\{h, c_+, \varphi_+(t_{k_0})\}$ for a fixed non-physics parameters such as k_0 and f.[14] Eqs. (238) and (240) put a bound of

$$c_+(\text{for max } H) < 1.78 \tag{241}$$

corresponding to $n < 2.6$. In this c_+ range, the c_+ dependence of Eq. (235) is weak. From Eq. (235), we thus conclude that the maximum H that is allowed by the present scenario is

$$H \lesssim 5 \times 10^{12}\ \text{GeV}\,(R_s(c_+))^{-50/219}\left(\frac{\alpha_{k_0}}{3\times 10^{-2}}\right)^{50/219}\left(\frac{\varphi_+(t_{k_0})/M_p}{10^{-1}}\right)^{100/219}. \tag{242}$$

Next, we still need to impose the constraints of Eq. (218) from the quantization. This constraint requires H to be large and φ_+/M_p to be small. We can thus denote this constraint as a lower bound on H:

$$\frac{0.82}{f} \times \left(3\times 10^{-3}\right)^{-\frac{3}{2}+\nu+O(\varepsilon)g_0(m/H)}\left(\frac{c_+}{0.1}\right)\left(\frac{\varphi_+(t_{k_0})/M_p}{0.1}\right) 8.7\times 10^{11}\ \text{GeV} < H. \tag{243}$$

[14] The choice of k_0 does contain phenomenological information in where the data constraints lie because one usually wants to choose k_0 where the data is accurate.

For any fixed c_+, both Eqs. (242) and (243) can be satisfied if $\varphi_+(t_{k_0})/M_p$ is small enough. With the right hand side of Eq. (242) set equal to the left hand side of Eq. (243), we can find another constraint on maximum $\varphi_+(t_{k_0})/M_p$ similar to that coming from Eq. (239). This latter constraint usually is more important than Eq. (239) for a blue spectral index. We thus arrive at the maximum H shown in Fig. 4 for the blue spectral index scenario considered in this section. The corresponding tensor-to-scalar power spectra ratio r is also shown on the right axis of the same figure. It is clear that inflationary scenarios consistent with a very blue isocurvature spectra (e.g. $n \gtrsim 2.0$) will not generate tensor spectra that is observable in the near future. Conversely, a tensor-to-scalar ratio at the level of $r = O(10^{-1})$ will disfavor this class of models.

Let us now summarize the how one obtained the upper bound on H. One minimizes the angle $\theta_+(t_{k_0})$ consistent with the validity of the classical equations of the motion and subject to the decoupling constraints. Putting this into the isocurvature formula and minimizing the isocurvature to curvature ratio by varying F_a/H subject to dark matter abundance constraints lead to Eq. (242) bound on H. The decoupling constraints are not particularly fundamental, but the study of that region is beyond the scope of this paper. Hence, even higher values of the tensor-to-scalar ratio may be valid with blue isocurvature perturbations, but the phenomenological signatures will be more complicated than the simple situation presented here.

3.2. Do Dressing Effects Give a Lower Bound on the Blue Isocurvature Spectrum?

From the definition of S_χ in Eq. (17), we would naively expect

$$\langle S_\chi S_\chi \rangle \sim 4\langle \frac{\delta\chi^{(N)}}{\chi_0}\frac{\delta\chi^{(N)}}{\chi_0}\rangle + 4\langle \zeta\zeta\rangle \left(\frac{\dot{\chi}_0}{H\chi_0}\right)^2 + \text{cross terms} \tag{244}$$

where the second term arises from the "dressing" effect of the isocurvature coming from the fact that spectator isocurvature is a contrast between the isocurvature field and the adiabatic field. For scale invariant isocurvature spectra (such as massless axions), we have $\dot{\chi}_0 = 0$ which means that this second term coefficient is negligible. However, for a blue spectrum, this coefficient is of order unity. Hence, one would naively expect

$$\langle S_\chi S_\chi \rangle \gtrsim \langle \zeta\zeta \rangle \quad \text{naive operator product expectation for blue spectra} \tag{245}$$

independently of $\langle\delta\chi^{(N)}\delta\chi^{(N)}\rangle/\chi_0^2$ amplitude.[15] Hence, a little puzzle arises whether one can conclude

$$\frac{\Delta_{s_\chi}^2}{\Delta_\zeta^2} \gtrsim 1 \tag{246}$$

for a blue isocurvature spectrum in the limit $H \to 0$ with $m/H = O(1)$ and $\chi_0(t_{k_0})$ fixed.

Theorem 3 trivially states that this lower bound with $\chi_0(t_{k_0})$ fixed does not exist since

$$\frac{\Delta_{s_\chi}^2}{\Delta_\zeta^2} \approx 1.3 \times 10^{-2} \times 2^{2\nu} \times |\Gamma(\nu)|^2 \left(\frac{\varepsilon}{10^{-2}}\right)\left(\frac{\chi_0}{M_p}\right)^{-2} \tag{247}$$

which vanishes as the inflaton slow-roll parameter $\varepsilon \to 0$ with $\langle\zeta\zeta\rangle$ fixed. That means that

$$\frac{\langle S_\chi S_\chi\rangle}{\langle\zeta\zeta\rangle} \to 0 \tag{248}$$

for blue spectra is possible if $\chi_0(t_{k_0})$ can be fixed contrary to the naive expectation from Eq. (245).[16] On the other hand, consistent quantization does give a lower bound shown in Eq. (139). However, Eq. (139) arises because $\chi_0(t_{k_0})/M_p$ cannot be fixed because of the quantization approximation used: i.e. the reason for Eq. (139) is different from the naive operator multiplication analysis. Indeed, $\Delta_{s_\chi}^2(k_0)$ shown in Eq. (63) is independent of ε_{k_0} (up to the inequalities implied by Eq. (69)) while $\Delta_\zeta^2(k_0)$ shown in Eq. (14) manifestly does depend on ε_{k_0}.

If one looks at the details of the proof to try to understand where the naive operator multiplication analysis goes wrong, one notes that there is a secular growth effect in which the isocurvature field $\delta\chi^{(G)}$ develops an adiabatic piece due to a secular superhorizon source effect. For example, in the spatially flat gauge, the gravitational interaction transmitted adiabatic piece is given by Eq. (93). The fact that this comes from gravitational physics can be understood from the fact that the off-diagonal mass squared in Eq. (82) are Planck suppressed, and these terms are responsible for Eq. (93). This can be interpreted as the effect of gravity imprinting dominant energy inhomogeneity information onto the subdominant isocurvature field. Hence, even if $\delta\chi^{(G)}$ at the quantum fluctuation level does not have ζ correlation information[17], it will on far superhorizon scales look

[15] This does not by itself give a bound on the total isocurvature perturbations which depend on the dark matter fraction ω_χ .

[16] This limit $\varepsilon \to 0$ can be fraught with strong coupling issues in the density perturbation computation formalism. We can neglect these issues and can take this limit formally since the point is that it decreases towards zero and not about the absolute magnitude.

[17] In the proof of theorem 3, the spatially flat subhorizon modes are essentially decoupled from the inflaton modes by χ_0/M_p. This allows one to determine the Bunch-Davies state quantum correlator independently of the sourced mixing with the inflaton in the subhorizon region.

like a mixture of adiabatic and nonadiabatic field, in precisely the combination to eliminate the ζ dependent pieces in $\langle S_\chi S_\chi \rangle$.

4. SUMMARY AND CONCLUSION

In this paper, we have presented three theorems and related corollaries concerning blue spectra produced by linear spectator isocurvature fields that give rise to CDM-photon isocurvature perturbations. Theorem 1 defines a superhorizon conserved quantity for systems possessing an approximate symmetry of $V_\chi'(\delta\chi) \approx V_\chi''(\delta\chi)\delta\chi$. The merit of this theorem compared to previous discussions of this topic in the literature is its ability to go beyond the end of inflation and the reheating process. Theorem 2 describes under what averaging conditions that fluid quantities behave as $\delta\chi_{nad}/\chi_0$. This second theorem merely restates what is known in the literature (see e.g. [91, 92]) in the context of current theorems. Theorem 3 describes the computation of the quantum isocurvature perturbations. The merit of theorem 3 compared to the previous discussion in the literature is the explicit canonical quantization in the presence of linearized gravitational constraints. The validity regime of this theorem imposes a nontrivial constraint of Eq. (138). If this condition is violated, the amplitude of quantization is expected to be more complicated than the simple analytic treatment presented here.

In Sec. 3.1, we have applied the theorems to the work of [1] and improved their computation. In the process, we have uncovered a conserved Noether current associated with $U(1)_{PQ}$ that is leading to the tracking of the axion field with the radial direction field. The final spectral formula is given in Eq. (212). The general magnitude comparison of the isocurvature blue part of the spectrum is shown in Fig. 1. The spectral break features and the validity of analytic computations were explored in Figs. 2 and 3. The maximum tensor-to-scalar ratio for which this simple scenario remains valid is shown in Fig. 4.

In Sec. 3.2, we have applied the theorems to explain how naive operator product estimates for the isocurvature correlator amplitude lower bound fails. The main physics is that the spectator field attains the inhomogeneities associated with the inflaton through its gravitational coupling. From a perturbation theory perspective, this inhomogeneity is attained through a secular effect which would naively be dropped from the consideration of perturbative expansion coefficient alone. From a physical perspective, the spectator field which undergoes no appreciable quantum fluctuations by themselves still attains an inhomogeneity that looks like the inflaton's inhomogeneities. Inter-

estingly, we do uncover in this paper an isocurvature correlator amplitude lower bound Eq. (139) whose phenomenological validity requires the dark matter fraction ω_χ to be much smaller than unity. If this lower bound is violated, the quantization of the isocurvature perturbations do not take on the simple form presented in this paper.

There are many possible future extensions of this work. Regarding the general applicability of the theorem, it would be interesting to find interaction strength boundaries for classes of models for which the linear spectator behavior of the isocurvature perturbation survives. Regarding the scenario of [1], we have explicitly laid out where the analytic computation fails near the break region of the spectra. Although we would naively expect that either side of the break region to be smoothly connected, we would also naively expect features to exist in that region. Numerical investigations of the features may be interesting for discovery potential. Other obvious future investigation possibilities include improving our understanding of the experimental discovery prospects of the blue spectral isocurvature perturbations.

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Appendix A: Particular Solution In the Subhorizon Region

Consider the approximate equation of motion object for $\delta\chi^{(sf)}$ coming from Eq. (78)

$$E \equiv \frac{d^2\delta\chi^{(sf)}}{dt^2} + 3H\partial_t\delta\chi^{(sf)} + \left(\frac{k^2}{a^2} + M_{22}^2\right)\delta\chi^{(sf)} + m^2\frac{\dot{\varphi}_0\chi_0}{M_p^2 H}\delta\varphi^{(sf)} \tag{A1}$$

where we have approximated

$$M_{21}^2 = m^2\frac{\dot{\varphi}_0\chi_0}{M_p^2 H}\left[1 + O\left(\frac{\chi_0}{M_P}\right)\right]. \tag{A2}$$

Without taking the superhorizon limit, consider the following particular solution ansatz:

$$\delta\chi^{(sf)} = \frac{\dot{\chi}_0}{\dot{\varphi}_0}\delta\varphi^{(sf)}. \tag{A3}$$

The equation of motion object E becomes

$$E = \frac{\dot{\chi}_0}{\dot{\varphi}_0}\left(\partial_t^2\delta\varphi + \left[3H - 2m^2\frac{\chi_0}{\dot{\chi}_0} + 2\frac{V'_\varphi(\varphi)}{\dot{\varphi}_0}\right]\delta\dot{\varphi} + W\delta\varphi\right) \tag{A4}$$

$$W \equiv \frac{k^2}{a^2} + M_{22}^2 - m^2 - 6m^2\frac{\chi_0}{\dot{\chi}_0}H + \frac{m^2}{M_p^2}\frac{\chi_0\dot{\varphi}_0^2}{H\dot{\chi}_0} + 6H\frac{V'_\varphi(\varphi_0)}{\dot{\varphi}_0} - 2\frac{m^2\chi_0 V'_\varphi(\varphi_0)}{\dot{\chi}_0\dot{\varphi}_0} + 2\frac{[V'_\varphi(\varphi_0)]^2}{\dot{\varphi}_0^2} + V''_\varphi(\varphi_0). \tag{A5}$$

Next, the usual slow-roll approximation gives

$$2\frac{V'_\varphi(\varphi)}{\dot{\varphi}_0} \approx -6H \tag{A6}$$

$$-2m^2\frac{\chi_0}{\dot{\chi}_0} \approx 6H \tag{A7}$$

which implies

$$W \approx \frac{k^2}{a^2} + M_{22}^2 - m^2 + (3\eta_V - 6\varepsilon)H^2. \tag{A8}$$

Note that Eq. (A7) assumes a small slow-roll factor analogous to ε but for the χ_0 field. We will call these expansion parameters

$$\eta_\chi \equiv \frac{m^2}{9H^2} \ll 1 \quad \varepsilon_\chi \equiv \frac{1}{54}\frac{m^4}{H^4}\frac{\chi_0^2}{M_p^2} \ll 1 \tag{A9}$$

whose motivation is detailed in Sec. C. Explicitly, one can show

$$-2m^2\frac{\chi_0}{\dot{\chi}_0} + 2\frac{V'_\varphi(\varphi)}{\dot{\varphi}_0} = \left[O(\varepsilon^{3/2}) + O(\sqrt{\varepsilon}\eta_V) + O(\sqrt{\varepsilon}\eta_\chi)\right]H \tag{A10}$$

Next. we know

$$M_{22}^2 = m^2[1 + O\left(\frac{\chi_0^2}{M_P^2}\right)] \tag{A11}$$

which yields

$$W \approx \frac{k^2}{a^2} + (3\eta_V - 6\varepsilon)H^2. \tag{A12}$$

Finally, we also write down the equation of motion for $\delta\varphi^{(sf)}$ as

$$\frac{d^2\delta\varphi^{(sf)}}{dt^2} + 3H\partial_t\delta\varphi^{(sf)} + \left(\frac{k^2}{a^2} + (3\eta_V - 6\varepsilon)H^2\right)\delta\varphi^{(sf)} + m^2\frac{\dot{\varphi}_0\chi_0}{M_p^2 H}\delta\chi^{(sf)} = 0 \tag{A13}$$

and note that

$$m^2\frac{\dot{\varphi}_0\chi_0}{M_p^2 H}\delta\chi^{(sf)} = m^2 \mathrm{sgn}\dot{\varphi}_0\frac{\sqrt{2\varepsilon}\chi_0}{M_p}\delta\chi^{(sf)} = O\left(\frac{\chi_0}{M_p}\right)\delta\chi^{(sf)}. \tag{A14}$$

We thus conclude

$$E = 0 + O\left(\frac{\chi_0}{M_p}\right) + O\left(\varepsilon^{n>1}\right) + O(\eta_\chi\sqrt{\varepsilon}) \tag{A15}$$

which means that Eq. (A3) solves the equation of motion even in the subhorizon region in the leading approximation.

Finally, note that the particular solution itself is of order

$$\frac{\dot{\chi}_0}{\dot{\varphi}_0}\delta\varphi^{(sf)} = O\left(\frac{\chi_0}{M_p}\right)\delta\varphi^{(sf)} \tag{A16}$$

which means that since we are dropping the same order in Eq. (A15), one might naively think there is no content in the solution. However, note that on the left hand side of Eq. (A1), there is a factor of $(k/a)^2$ which makes this term non-negligible. The point of this section was to show that such unsuppressed terms are all canceled by the slow-roll equations of motion to leading order in slow-roll expansion.

Appendix B: Mixture

Result of theorem 3 is applicable only when the dark matter χ can be made to be totality of dark matter. Suppose

$$\frac{\delta\rho_{cdm}}{\rho_{cdm}} = \frac{\delta\rho_X + \delta\rho_Y}{\rho_X + \rho_Y} \tag{B1}$$

where X can be the χ particle and Y is the rest of the cold dark matter. We can rewrite this as

$$\begin{aligned}\frac{\delta\rho_{cdm}}{\rho_{cdm}} &= \frac{\rho_X\delta_X + \rho_Y\delta_Y}{\rho_X + \rho_Y} && \text{(B2)}\\ &= \omega_X\delta_X + \omega_Y\delta_Y. && \text{(B3)}\end{aligned}$$

where

$$\omega_X \equiv \frac{\rho_X}{\rho_X + \rho_Y} \qquad \omega_Y \equiv \frac{\rho_Y}{\rho_X + \rho_Y} \tag{B4}$$

such that $\omega_X + \omega_Y = 1$. Hence

$$\begin{aligned}\delta_S &= \frac{\delta\rho_{cdm}}{\rho_{cdm}} - \frac{3}{4}\frac{\delta\rho_\gamma}{\rho_\gamma} && \text{(B5)}\\ &= \omega_X\delta_X + \omega_Y\delta_Y - \frac{3}{4}\delta_\gamma && \text{(B6)}\end{aligned}$$

If

$$\delta_Y = \frac{3}{4}\delta_\gamma, \tag{B7}$$

then

$$\delta_S = \omega_X\left[\delta_X - \frac{3}{4}\delta_\gamma\right] = \omega_X\delta_{S_X} \tag{B8}$$

where δ_{S_X} is the isocurvature in X component. Hence, with mixing, the isocurvature is diluted by a factor ω_X.

Appendix C: Slow-Roll of χ

In this section, we motivate a slow-roll expansion parameter for χ field. Consider

$$\ddot{\chi}_0 + 3H\dot{\chi}_0 + V'_\chi(\chi_0) = 0 \tag{C1}$$

where H is an externally determined time dependent function. Divide through by HM_p^2.

$$\frac{\ddot{\chi}_0}{HM_p^2} + 3\frac{\dot{\chi}_0}{M_p^2} + \frac{V'_\chi(\chi_0)}{HM_p^2} = 0 \tag{C2}$$

Define a formal hierarchy

$$\frac{\ddot{\chi}_0}{HM_p^2} \ll \left\{3\frac{\dot{\chi}_0}{M_p^2}, \frac{V'_\chi(\chi_0)}{HM_p^2}\right\} \tag{C3}$$

and expand $\dot{\chi}_0/M_p^2$ about zero using a formal perturbation parameter λ:

$$\{\frac{\dot{\chi}_0}{M_p^2} = O(\lambda), \;\; \frac{V'_\chi(\chi_0)}{HM_p^2} = O(\lambda)\} \tag{C4}$$

Using this expansion, construct a trial solution

$$\dot{\chi}_0 = -\frac{V'_\chi(\chi_0(t))}{3H(t)}\lambda + v_1\lambda^2. \tag{C5}$$

Put the trial solution Eq. (C5) into Eq. (C1). For this endeavor, we need to evaluate $\ddot{\chi}_0$ to second order in λ:

$$\begin{aligned}
\ddot{\chi}_0 &= -\frac{V''_\chi(\chi_0(t))}{3H(t)}\dot{\chi}_0\lambda + \frac{V'_\chi(\chi_0(t))}{3H^2}\dot{H}\lambda + \lambda^2\dot{v}_1 && \text{(C6)}\\
&= \frac{V''_\chi(\chi_0(t))}{3H(t)}\frac{V'_\chi(\chi_0(t))}{3H(t)}\lambda^2 + \frac{V'_\chi(\chi_0(t))}{3H^2}\dot{H}\lambda + \lambda^2\dot{v}_1 + O(\lambda^3) && \text{(C7)}
\end{aligned}$$

To finish expanding the right hand side, we need to evaluate $\dot{H}$. According to the usual slow-roll, we have

$$\dot{H} = -\varepsilon H^2. \tag{C8}$$

Hence, we conclude

$$\ddot{\chi}_0 = \frac{1}{9H^2}V'_\chi(\chi_0)V''_\chi(\chi_0)\lambda^2 - \lambda\varepsilon\frac{V'_\chi(\chi_0)}{3} + \lambda^2\dot{v}_1 + O(\lambda^3) \tag{C9}$$

Put the trial solution Eq. (C5) into Eq. (C2), account for the second term of Eq. (C4), treat $\varepsilon\lambda = O(\lambda^2)$ in the formal counting, and collect $O(\lambda)$ and $O(\lambda^2)$:

$$O(\lambda^2): \;\; \frac{1}{9H^2}V'_\chi(\chi_0)V''_\chi(\chi_0) - \frac{\left[V'_\chi(\chi_0)\right]^3}{54H^4M_p^2} + \dot{v}_1 + 3\frac{\sqrt{V_\chi}}{\sqrt{3}M_p}v_1 = 0. \tag{C10}$$

This can be solved

$$v_1 = e^{-3\int dtH}\int dt\frac{\left[V'_\chi(\chi_0)\right]^3}{54H^4M_p^2}e^{3\int dtH} - e^{-3\int dtH}\int dt\frac{1}{9H^2}V'_\chi(\chi_0)V''_\chi(\chi_0)e^{3\int dtH}. \tag{C11}$$

Hence, we conclude the fractional correction to $\dot{\phi}$ is

$$\frac{v_1}{\frac{V'_\chi(\chi_0(t))}{3H(t)}} = O\left(\frac{1}{\frac{V'_\chi(\chi_0(t))}{3H(t)}}\times\frac{1}{3H}\times\frac{1}{9H^2}V'_\chi(\chi_0)V''_\chi(\chi_0)\right) + O\left(\frac{1}{\frac{V'_\chi(\chi_0(t))}{3H(t)}}\times\frac{1}{3H}\times\frac{\left[V'_\chi(\chi_0)\right]^3}{54H^4M_p^2}\right). \tag{C12}$$

This motivates us to define the following slow-roll parameters:

$$\frac{1}{\frac{V'_\chi(\chi_0)}{3H}}\times\frac{1}{3H}\times\frac{1}{9H^2}V'_\chi(\chi_0)V''_\chi(\chi_0) \;=\; \frac{1}{9H^2}V''_\chi(\chi_0) \equiv \eta_\chi \tag{C13}$$

$$\frac{1}{\frac{V'_\chi(\chi_0(t))}{3H(t)}}\times\frac{1}{3H}\times\frac{\left[V'_\chi(\chi_0)\right]^3}{54H^4M_p^2} \;=\; \frac{\left[V'_\chi(\chi_0)\right]^2}{54H^4M_p^2} \equiv \varepsilon_\chi. \tag{C14}$$

Obviously, this is not unique, and other slow-roll definitions exist. See e.g. [99].

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